8.1 Percent, Sales Tax, Discounts

Percentages express fractions in terms of parts per 100 (Latin: *per centum*, per hundred).

Example. Of 583 surveyed inhabitants of a certain city, 305 said they approve of the job the mayor is doing. What percent of respondents approve of the job the mayor is doing?

The formula $A = P \cdot B$ states that A is P percent of B, where P is expressed as a decimal number.

Example. On a retail website, 23% of purchases had gift wrap requested. If there were 1,380 orders, how many had gift wrap requested?

Example. 7 is 38% of what number?

Example. A toaster costs \$14.99. If you bought the toaster in Kentucky, where sales tax is 6%, what is the total cost of the toaster?

Example. A cell phone costs \$139, but it was discounted 20%. How much does it cost now?

Example. Average US gas price was \$2.42 in January of 2021, and \$3.23 in July of 2021. What percent increase is this?

8.2 Income Tax

In the computation of income tax, several common terms are used.

Gross income:	All the money a taxpayer received during the year, typically: wages, interest, lottery winnings, etc.
Adjusted gross income:	Gross income – adjustments
Taxable income:	Adjusted gross income – (Exemptions + Deductions)
Adjustments:	typically contributions to retirement accounts
Exemptions:	amounts set by law to be not taxable, per person
Itemized Deductions:	expenses for particular categories recognized by law, typically: mortgage interest, state income and property taxes, charitable contributions, large medical expenses
Standard Deduction:	expense allowed for everyone, amount set by law

Note: Deductions used for taxable income is the larger of the standard deduction or the sum of itemized deductions.

Example. In 2016, single woman Sarah's earnings, adjustments, deductions and exemptions are described below.

Gross income:	wages \$55,300, interest \$431
Adjustments:	contributions to a retirement account \$2000
Exemptions:	\$4050
Itemized Deductions:	mortgage interest \$6,000 property tax \$1500 state income tax \$1700 charitable contribution \$400
Standard Deduction:	\$6300

Compute Sarah's adjusted gross income and taxable income.

Federal income tax in the U.S. is paid on taxable income. Income taxes are paid on different parts of a person's taxable income at different rates, called *marginal rates*.

Every taxpayer has one of four filing statuses, which determines the brackets of income that tax rates apply to.

The table shows 2016 marginal tax rates, standard deductions and exemptions by filing status.

Tax rate	Single	Married Filing Separately	Married Filing Jointly	Head of Household
10%	up to $9275	up to $9275	up to $18,550	up to $13250
15%	$9276 to $37,650	$9276 to $37,650	$18,551 to $75,300	$13,251 to $50,400
25%	$37,651 to $91,150	$37,651 to $75,950	$75,301 to $151,900	$50,401 to $130,150
28%	$91,151 to $190,150	$75,951 to $115,725	$151,901 to $231,450	$130,151 to $210,800
33%	$190,151 to $413,350	$115,726 to $206,675	$231,451 to $413,350	$210,801 to $413,350
35%	$413,351 to $415,050	$206,676 to $233,475	$413,351 to $466,950	$413,351 to $441,000
39.6%	above $415,050	above $233,475	above $466,950	above $441,000
Standard Deduction	$6300	$6300	$12,600	$9300
Exemptions (per person)	$4050	$4050	$4050	$4050

Example. Based on table, compute Sarah's income tax.

8.3 Simple Interest

When we deposit money in a bank account the bank pays us interest. When we take out a loan from a bank, we pay interest on the loan.

Simple interest is computed using the formula $I = Prt$.

Example. You deposit \$1,000 in an account earning 4% simple interest. How much interest is earned in 9 months?

If all funds were withdrawn after 9 months, how much was in the account at the time of withdrawal?

Simple interest future value formula is $A = P(1 + rt)$.

Note: Formula has four variables. If we know three of them, we can solve for the fourth.

Example. How much needs to be deposited today in an account paying 3.75% simple interest annually, so that the account is worth \$3,000 in 4 years?

Example. A pawn shop loaned Rick \$130 which he has to repay with simple interest in one month. If he paid back \$156 at the end of the month, what is the annual interest rate?

Mathematical Concepts — Lecture notes
MAT 117, Fall 2021 — D. Ivanšić

8.0 Review of Powers and Roots

Review of powers and roots

$5^4 =$ $\qquad$ $x^n =$

$5^{-3} =$ $\qquad$ $x^{-n} =$

Example. Compute the following:

$4^3 =$ $\qquad$ $10^4 =$

$2^{-7} =$ $\qquad$ $10^{-3} =$

Roots: $\sqrt[n]{a}$ is the answer to the question $?^n = a$.

Example. Find the roots.

$\sqrt{25} =$ $\qquad$ $\sqrt{49} =$ $\qquad$ $\sqrt[3]{8} =$

$\sqrt[3]{125} =$ $\qquad$ $\sqrt[4]{81} =$ $\qquad$ $\sqrt[5]{32} =$

Roots can also be written as fractional exponents $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Example. Write root as a fractional exponent.

$\sqrt{33} =$ $\qquad$ $\sqrt[3]{7} =$ $\qquad$ $\sqrt[4]{15} =$ $\qquad$ $\sqrt[5]{12} =$

An important rule for exponents: $(x^a)^b = x^{ab}$

Example. Simplify.

$(5^2)^4 =$ $\qquad$ $(3^7)^{-2} =$ $\qquad$ $(11^4)^{\frac{1}{2}} =$ $\qquad$ $(7^{12})^{\frac{1}{12}} =$

Example. Solve the equations.

$x^{14} = 3$

$(1 + y)^{18} = 1.25$

$100(1 + r)^{24} = 180$

8.4 Compound Interest

Suppose \$500 is deposited in an account earning 4% annually.

After 1 year, this much is in the account: $A_1 =$

Now suppose interest is added to the account at the end of the year. The following year, interest is computed on the new principal, and interest is added to the account at the end of the year again.

After 2 years, this much is in the account: $A_2 =$

After 3 years, this much is in the account: $A_3 =$

After 4 years, this much is in the account: $A_4 =$

Note: Amount after 1 year = current amount $\cdot$ 1.04

After 7 years, this much is in the account: $A_7 =$

After n years, this much is in the account: $A_n =$

Interest computed in this way is called *compound interest*, as interest is collected on previous interest and principal.

Example. Compare the future value after 7 years of \$500 deposited in an account earning 4% using compound and simple interest. Which is better?

Now suppose \$500 is deposited in an account earning 4% annually, but interest is added every three months ("interest is compounded every three months").

After 3 months, this much is in the account: $A_{3m} =$

Note: Like before, amount after 3 months = current amount · 1.01

After 6 months, this much is in the account: $A_{6m} =$

After 9 months, this much is in the account: $A_{9m} =$

After 1 year, this much is in the account: $A_1 =$

After 2 years, this much is in the account: $A_2 =$

After 3 years, this much is in the account: $A_3 =$

Thus we get: amount after a number of periods of compounding is:

$$A = P(1 + \text{periodic rate})^{\text{number of periods}}$$

Let	r = annual interest rate	so we get	periodic rate $= \dfrac{r}{n}$
	n = number of periods per year		number of periods $= nt$

Compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = future value of account

P = principal

t = time in years

n = number of times interest is compounded per year

r = annual interest rate

Note: for this formula, principal is deposited at the beginning of t years, and no additional money — aside from interest — is deposited into the account.

Example. Find the value of a \$700 investment after 4 years if it earns 3.25% interest, compounded monthly. How much interest was earned overall?

Example. Sheila would like to invest some money to buy a \$6,000 used car in three years. How much does she need to deposit if she can get 5% interest, compounded monthly?

Example. What is a better deal: earning 4.05% compounded semiannually, or earning 3.95% compounded daily?

The actual percentage increase after one year is called *annual percentage yield (APY)* or *annual effective yield*, and is computed by the formula:

$$APY = \left(1 + \frac{r}{n}\right)^n - 1$$

As one would expect, the more times interest is compounded in a year (the larger the n), the greater the amount in the account at the end of one year. This does not grow without bound, however: already for $n \geq 12$ the difference in the amount is small as n increases. If interest is compounded every moment — the term is *continuously* — then the compound interest formula turns into the one below.

Compound interest formula for continuous compounding.

$$A = Pe^{rt}$$

r = annual interest rate $\qquad$ A = future value of account $\qquad$ P = principal

Example. If \$10,000 was deposited into an account bearing 8%, compare the amount in the account after one year if it is compounded daily versus continuously.

Example. What (annual) interest rate is needed so an account compounded quarterly doubles in 5 years?

8.5 Systematic Savings: Annuities

Saving is seldom done as one-time large deposits. Rather, smaller amounts are usually deposited over time.

Example. At the end of every quarter, you deposit \$500 into an account bearing 6% interest, compounded quarterly. How much is in the account at the end of the year?

If the saving is done for two years, how much is in the account after two years?

We see that the expression of form $a^{k-1}+a^{k-2}+a^{k-3}+\cdots+a^2+a+1$ comes up. There is a formula that simplifies its evaluation:

$$a^{k-1}+a^{k-2}+a^{k-3}+\cdots+a^2+a+1=\frac{a^k-1}{a-1}$$

Verify it for $k=4$:

In the example from the beginning, we get:

one-year savings =

two-year savings =

An annuity is a sequence of equal payments at equal time intervals into an account.

Annuity Formula

$$A=P\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}$$

A = future value of account

P = amount deposited at *end* of every compounding period

t = time in years

n = number of times interest is compounded per year

r = annual interest rate

Example. After your child is born, you deposit \$50 at the end of every month into an account bearing 5% interest compounded monthly.
a) How much is in the account after 18 years?
b) How much of the future value is from contributions, and how much from interest?

Example. How much needs to be deposited at the end of every year into an account bearing 7% compounded annually, in order to have \$100,000 in 20 years? How much of the \$100,000 is from interest?

8.6, 8.7 Amortized Loans

Example. \$20,000 is borrowed to buy a new car at 6% interest, compounded monthly. If the loan is repaid in 5 years with equal payments, what is the monthly payment PMT?

The payment is figured using the following rule: the future value of the series of payments PMT must be the same as the future value of \$20,000 (both in five years).

Loan formula

$$P = PMT\,\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}$$

P = amount borrowed

t = time in years

PMT = amount paid at *end* of every compounding period

n = number of times interest is compounded per year

r = annual interest rate

Example. Suppose you can afford at most \$700 toward payment of a 30-year mortgage at 4.125% interest, compounded monthly. What is the largest loan that you can finance with this payment?

Example. Now assume you took out a 30-year loan for the above amount.
a) What is the total amount of your payments? How much of this is for the interest?
b) How much do you owe after 16 years of payments?
(Method: amount owed = present value of *remaining* payments).

How amortized loans work: If you owe a balance on a loan at the beginning of the month, interest accrues for the month. At the end of the month, it is added to the loan. At the same time, you make a payment, which pays for the interest, and a little extra, which gets subtracted from the loan balance. Clearly, the payment must be greater than the interest accrued, or you'd never pay off the principal!

Write the amortization schedule for the loan in the previous example for
a) the first three payments,
b) the two payments coming after 16 years.

Example. A certain pink bunny buys a drum for \$729 on a 1-year loan with interest rate 8%, compounded quarterly. Write the amortization schedule for the loan.