Mathematical Concepts — Lecture notes
MAT 117, Fall 2021 — D. Ivanšić

13.1 Voting Methods

We explore various methods of voting, based on ranking of the choices, which captures most real-life voting methods.

Plurality Method: Each voter votes for one candidate, and the winner is the candidate that receives the most votes.

Plurality with Elimination Method: Each voter votes for one candidate. If a candidate receives a majority of the votes (>50%), that candidate is the winner. If no candidate receives a majority, the candidate with fewest votes (more, if there is a tie) is removed and the remaining candidates compete in a new election. Repeat until only two canditates remain, or there is a majority winner.

To save the cost of running a new election after an elimination, one can go with ranking the candidates and inferring the elimination election results from the rankings. We study voting methods based on rankings.

Example. Eleven people in a depatment are deciding on which day to schedule their weekly meetings. The preference rankings are below. Find which option wins under plurality and plurality with runoff methods.

Votes	1	3	1	1	1	2	2
1st	T	T	W	W	Th	Th	F
2nd	W	F	Th	F	T	W	W
3rd	F	W	F	T	F	T	Th
4th	Th	Th	T	Th	W	F	T

Example. Find the winner in the election performed in class (ranking the choices) under these two rules.

Borda Count Method: Voters rank the candidates from the most favorable to the least favorable. Last place votes are awarded one point, next-to-last-place votes are awarded two points, and so on. The candidate receiving the most points wins.

Example. People were asked to rank their favorite ice cream flavors among chocolate, strawberry and vanilla. The results (percentages of respondents) are in the table. Find which option is most popular using the Borda count, plurality, and plurality with runoff methods.

Votes	3	33	7	27	10	20
1st	C	C	S	S	V	V
2nd	S	V	C	V	C	S
3rd	V	S	V	C	S	C

Pairwise Comparison Method: Voters rank the candidates and every candidate is compared with every other candidate. For every contest that a candidate wins, the candidate receives 1 point. In contests where candidates tie, both receive $\frac{1}{2}$ point. The winner is the candidate with the most points.

Example. An AM station seeks to merge with a partner. An FM station, a TV station and a newspaper have submitted offers. The AM station's stockholders ranked the potential partners as follows (percentages). Which one, if any, wins the pairwise-comparison method?

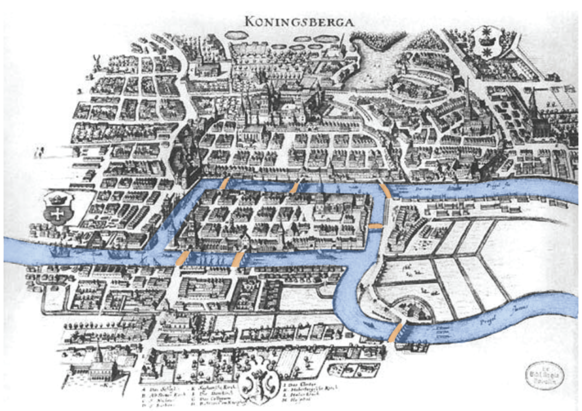
Votes	16	17	20	9	11	27
1st	FM	FM	TV	TV	N	N
2nd	TV	N	FM	N	FM	TV
3rd	N	TV	N	FM	TV	FM

Example. Which candidate wins in the election performed in class under the Borda count and pairwise comparison methods?

14.1, 14.2 Graphs, Euler's Theorem

Example. If an airline wanted to display which cities it flies to, it might use a picture like this. (Two cities are connected if there is a direct flight between them.)

Example. The picture below is the map of the city of Königsberg in the early 1700s (Kaliningrad today). Its residents were wondering if it is possible to walk around the town so that you cross every bridge exactly once (you do not necessarily have to finish where you started).



Example. Create a simple picture that models which of these states share a border.

The previous examples were abstracted using a mathematical concept called a graph, consisting of vertices (points) and edges (line segments connecting them). As a matter of fact, the Königsberg residents asked the German mathematician Euler [Oi'ler] to solve this problem, which he did, formulating the first theorem of the branch of mathematics called **graph theory**.

Terminology:

Vertices are **adjacent** if there is an edge connecting them.

A **path** is a sequence of adjacent vertices and edges connecting them (ignore book: "edge is in a path only once").

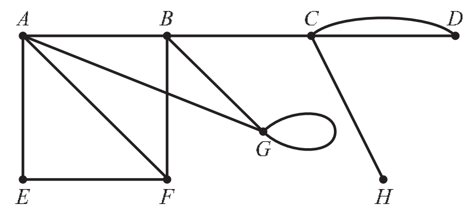
A **circuit** is a path that starts and ends with the same vertex.

The graph is **connected** if every two vertices can be joined by a path.

An edge is a **bridge** if its removal makes the graph disconnected.

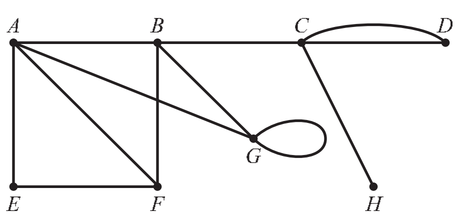
Degree of a vertex is number of edges at that vertex (loops count as two).

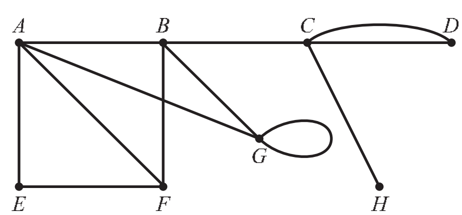
Graphs are **equivalent** if they have the same number of vertices connected to each other in the same way. Practically, this means you can deform one to another without breaking the edges.



Definition. An **Euler path** is a path that passes every edge of the graph exactly once. An **Euler circuit** is an Euler path that starts and ends at the same vertex (so it is a circuit).

Example. Find an Euler path or circuit for the graph.

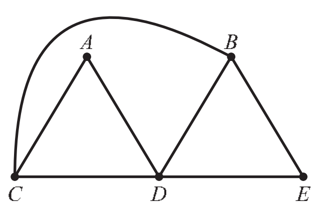


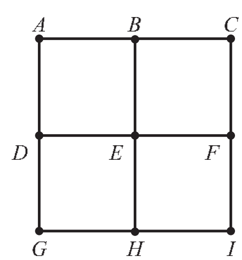


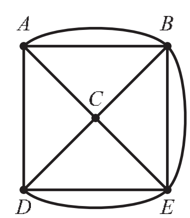
Theorem. (Euler's Theorem) For a connected graph:
1) If the graph has exactly two odd vertices, then it has at least one Euler path, but no Euler circuit. Every Euler path has to start and end at the odd vertices.
2) If the graph has no odd vertices, then it has an Euler circuit which can start and end at any vertex.
3) If the graph has more than two odd vertices, then it has no Euler paths nor circuits.

Why it's true: suppose there is an Euler path. Consider a vertex that is neither the start nor the end of the path. While going on the path, this vertex is entered and exited the same number of times and every edge adjacent to the vertex is used up. Therefore, the number of edges adjacent to this vertex must be even. This justifies 3, saying that if an Euler path exists, there can be no more than two odd vertices. Other parts are more complex.

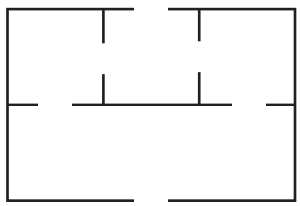
Example. Determine whether each of the following graphs has an Euler path or an Euler circuit. If it does, find it and state the order in which the vertices are visited, if not, explain why not.







Example. Can you walk through the house whose floor plan is pictured so that you pass through every door exactly once?



Example. An advertiser is leaving flyers in the neighborhood shown in the picture. Houses are on both sides of the street, and the advertiser always walks one row of houses on one side of the street at a time.
a) Draw a graph that models the neighborhood.
b) Can the advertiser deliver the flyer to every house in the neighborhood without walking by any row of houses twice and return to the starting point? If so, display the route.