11.1 The Fundamental Counting Principle

Example. Businessman Joe has 4 jackets and 3 pairs of pants to choose from. How many different outfits can he put together?

Example. Now suppose Joe also has 5 shirts to choose from. How many different outfits can he put together now?

Example. Kentucky license plates have 3 letters and 3 numbers. How many different license plates is it possible to make using those symbols?

How about if the middle letter is a vowel, and the first and third letter are consonants?

The Fundamental Counting Principle. The number of ways in which a sequence of things can occur is the product of the number of ways in which each one can occur.

Example. A coin is tossed three times. How many possible outcomes are there to this experiment?

Example. Two dice are rolled. How many possible outcomes are there to this experiment?

11.4 Fundamentals of Probability

Many events in life are unpredictable. Probability is a way of measuring how likely it is that an event will occur.

Probability of an event is a number between 0 and 1 (or 0% and 100%), where an event with probability 0 is impossible and an event with probability 1 is certain.

Example. The probability of:

a) Getting heads on a coin toss is:

b) Rolling a 3 on a single throw of a die is:

c) Drawing a queen from a deck of cards is:

d) Randomly choosing a female student from
a group of 13 female and 9 male students is:

The above examples were simple, and it was not hard to assign probability to them. For more complicated examples we need to introduce some language to set things up precisely.

Experiment = any occurence (action) for which the outcome is uncertain

Outcome = a possible result of the occurence

Sample space S = set of all occurences

Event E = subset of the sample space, that is, subset of the set of all occurences

Once the experiment is set up this way, then we compute probability of an event this way:

Theoretical Probability. If the sample space has $n(S)$ equally likely outcomes, then the theoretical probability of the event E, denoted $P(E)$, is

$$P(E) = \frac{\text{number of outcomes in event } E}{\text{total number of possible outcomes}} = \frac{n(E)}{n(S)}$$

Example. A bag contains balls numbered 1–17. A ball is drawn from the bag without looking. What is the probability that the number drawn is

a) less than 10 b) divisible by 3 c) has even sum of digits?

Example. A coin is tossed three times. What is the probability that we get

a) all heads b) exactly one head c) more heads than tails?

Example. Two dice are rolled. What is the probability that

a) the same number is on both dice b) the sum on the dice is 9

c) at least one of the numbers is 3?

Empirical probability is based on physical observations of how frequently an event occurs.

Empirical probability of an event E is

$$P(E) = \frac{\text{observed number of times } E \text{ occurs}}{\text{total number of observed occurences}}$$

Example. Over 10 years, Murray has had 2120 sunny days.

$P(\text{random day is sunny}) =$

Example. A small dog gets 120 grams of food in its bowl every day. We observe how much is left at the end of the day. Over 15 days, the measurements were:

$$16, 18, 21, 6, 7, 9, 0, 4, 17, 25, 40, 22, 23, 13, 2$$

$P(\text{between 10 and 20 grams are left}) =$

$P(\text{more than 30 grams are left}) =$

$P(\text{less than 10 grams are left}) =$

$P(\text{1 gram is left}) =$

Note: If an experiment can be assigned theoretical probability and can be physically observed, the more observations are carried out, the closer the empirical probability is to theoretical probability.

11.6 Events Involving Not and Or, Odds

Example. What is the probability of **not** getting a 5 on a roll of a die?

Complement rules of probability

$$P(\text{not } E) = 1 - P(E) \qquad\qquad P(E) = 1 - P(\text{not } E)$$

Example. A random card is drawn from a deck.

$P(\text{card is not a picture card}) =$

Example. What is the probability of getting sum 7 or 9 on a roll of two dice?

Definition. If events A and B cannot occur at the same time, they are said to be *mutually exclusive.*

Probability for mutually exclusive events. If events A and B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B)$$

Example. A random card is drawn from a deck. What is the probability that it is a picture card or a spade?

Probability for general events

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example. On a Friday evening, at a convenience store:

35% of customers bought potato chips or tortilla chips
22% of customers bought potato chips
20% of customers bought tortilla chips

What is that probability that a random customer bought potato chips and tortilla chips?

The rules of probability can be visualized. Probability works like area:

Odds in favor and against

Example. Two dice are rolled.

We get sum on dice less than 7 in _____ ways

We do not get sum on dice less than 7 in _____ ways

We say that: *odds in favor* of sum less than 7 are _____ to _____

odds against sum less than 7 are _____ to _____

In general, if there are: a outcomes in event E
b outcomes in event not E

We say that: odds in favor of E are a to b
odds against E are b to a.

Example. Suppose the odds against a horse winning a race are 3 to 2.
a) Is winning or not winning more likely?
b) What is the probability of the horse winning the race?

Odds to probability. If the odds in favor of event E are a to b, then

$$P(E) = \frac{a}{a+b} \qquad\qquad P(\text{not } E) = \frac{b}{a+b}$$

Example. What are the odds in favor and odds against getting two heads on two tosses of a coin?

Probability to odds

$$\text{odds in favor of } E = \frac{P(E)}{P(\text{not } E)} \qquad\qquad \text{odds against } E = \frac{P(\text{not } E)}{P(E)}$$

11.7 Events Involving And; Conditional Probability

Definition. Two events are *independent* if the occurence of one has no effect on the likelihood of the other.

Examples. Tossing a coin or rolling a die two or more times. What comes up on a subsequent toss or roll does not depend on previous tosses or rolls.

Drawing a card randomly from a deck several times as long as the card is returned each time and deck shuffled.

Note: If the card is not returned, then the second draw is not independent from the first one. For example, if the first card is a king, the probability of drawing a king on the second draw diminishes from 4/52, because now there are 3 kings left in the deck of 51 cards. If the first card is not a king, the probability of drawing a king increases, as there are now 4 kings among 51 remaining cards.

And probability with independent events

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example. A die is rolled three times. What is the probability we get
a) a 4 each time
b) at least one 4?

And Probability for general events

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given that } A \text{ has occured}) = P(A) \cdot P(B|A)$$

Example. Two socks are consecutively drawn from a bag with 18 black and 10 red socks. What is the probability that

a) both are red?
b) first is black and second is red?
c) we draw a black and a red sock?
d) second is black, if first one is red?
e) second is black if first one is black?
f) second is black?

Example. Two cards are drawn from a deck. What is the probability that:
a) they are both red?
b) they are both number cards?
c) all three are picture cards, if three are drawn?

Example. The following is a breakdown of people present at a hospital employee meeting:

	Men	Women
Nurses	10	21
Non-nurses	18	9

What is the probability that a randomly selected employee at the meeting:
a) is a nurse?
b) is a nurse, given that they are a woman?

$P(B|A)$ is called *conditional probability*. When we compute $P(B|A)$, we assume that the sample space is restricted to events in A.

A formula for conditional probability

$$P(B|A) = \frac{\text{number of outcomes common to } B \text{ and } A}{\text{number outcomes in } A} = \frac{n(A \cap B)}{n(A)}$$

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11.8 Expected Value

Example. Roll a die 50 times and record how many times you get each number:

Roll	1	2	3	4	5	6	sum
Times occured							

Suppose the rolls were part of a game of chance where you bet \$1. If you roll a winning number, you win \$5. Suppose the winning number is _____.

a) How much did you win (or lose) on your 50 tries?
b) Pool your results with 3 other students. How much did your group win (or lose)?
c) Pool your results with the whole class. How much did the whole class win (or lose)?
d) What is the average win per roll based on total class results?

	Times won	Times lost	Winnings
You			
Group			
Class			

Suppose an experiment results in a number. Repeating the experiment many times produces a sequence of numbers. "Expected value" captures the average of those numbers.

Definition. The expected value of an experiment is computed by multiplying the value A_i of each outcome with the probability P_i of that outcome and summing the results.

$$E = P_1A_1 + P_2A_2 + \cdots + P_nA_n$$

Example. Compute the expected value of the above game and compare it to the average winning per roll, computed above.

Note. In usual games of chance, the expected value is always negative (otherwise, in the long run, the organizer would lose). The "fair price" to play a game is the one you would charge in order to have expected value equal to zero. It is found by:

$$\text{Fair price} = \text{expected value} + \text{cost to play}$$

In the example above, what is the fair price?

Example. To play roulette a \$1 bet is made and a number 1–36 is chosen. The roulette wheel has slots 0, 00, 1–36. If the ball lands in a chosen slot, you get \$36. What is the expected value of this experiment? What is the fair price?

Example. A multiple choice test has 5 answers, one of which is correct. You get 1 point for a correct answer, 0 points for not attempting a question, and $1/4$ of a point is subtracted for an incorrect answer.

a) What is the expected value of a random guess?

b) If you can rule out two answers, what is the expected value of a random guess?