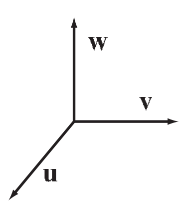
Calculus 3 — Exam 1
MAT 309, Spring 2021 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (11pts) Let $\mathbf{u} = \langle 1, 3, -1 \rangle$ and $\mathbf{v} = \langle 0, 2, 1 \rangle$.
a) Calculate $-2\mathbf{u}$, $3\mathbf{v} - 4\mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$.
b) Find a vector of length $\sqrt{5}$ in direction of $\mathbf{u}$.
c) If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, find $\cos\theta$.

2. (12pts) In the picture, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular and all have length 3.
a) Draw the vector $\mathbf{u} - \mathbf{v}$ with its tail coinciding with the other tails.
b) Which is longer (if any): $\mathbf{u} \times \mathbf{v}$ or $\mathbf{u} \times (\mathbf{u} - \mathbf{v})$?
c) Draw the vector $\mathbf{w} \times (\mathbf{u} - \mathbf{v})$. Accurate length is not important.



3. (8pts) Draw the set in
$\mathbf{R}^3$ described by:

$x^2 + y^2 + z^2 \geq 1$, $y = x$

4. (12pts) Find the equation of the plane that contains the lines given by parametric equations: $x = 1 + 2t$, $y = -2 - t$, $z = -3 + 4t$ and $x = 5 - t$, $y = -4 + 3t$, $z = 5 + t$. (These lines intersect — or they wouldn't determine a plane — but the point of intersection is not needed, so don't look for it.)

5. (16pts) This problem is about the surface $x^2 - 2y^2 + 5z^2 = 0$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

6. (14pts) The curve $\mathbf{r}(t) = \left\langle 2\cos t, 2\sin t, \frac{1}{3}\sin(4t) \right\rangle$ is given, t any real number.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = \frac{\pi}{2}$ and sketch the tangent line.

7. (13pts) The points $A = (1, 3, -2)$ and $B = (4, -1, 3)$ are given.
a) Write parametric equations of the line segment AB.
b) Compute the length of the line segment using the parametrization and arc length formula.
c) Compare your answer in b) with the distance from A to B.

8. (14pts) An arrow is launched from ground level at a 45° angle with initial speed 50 meters per second.
a) Assuming gravity acts in the usual negative y-direction (let $g = 10$), find the vector function $\mathbf{r}(t)$ representing the position of the arrow.
b) Find the range of the arrow.
c) Find the maximum height the arrow reaches.

Bonus (10pts) Find the parametric equations of the line that is the intersection of the planes $x - y + 2z = 2$ and $x - y - 3z = 6$.

Calculus 3 — Exam 2   **MAT 309, Spring 2021 — D. Ivanšić**

Name: ______________________________

Show all your work!

1. (10pts) Let $f(x,y)=\sqrt{y-x^2}$.
a) Find the domain of f.
b) Sketch the contour map for the function, drawing level curves for levels $k=-1,0,1,2$. Note the domain on the picture.
c) Suppose $f(x,y)$ is the temperature at point (x,y) and a heat-seeking insect (always moves in direction of greatest heat increase) starts at point $(1,2)$. Sketch the path the insect will take and explain.

2. (16pts) Let $f(x,y)=xe^{x^3+y^3}$.
a) At point $(1,0)$, find the directional derivative of f in the direction of $\langle -2,1\rangle$.
b) In what direction is the directional derivative the greatest, and what is its value?

3. (12pts) Consider the elliptical cone $y^2 + 3z^2 - x^2 = 0$.
a) Find the equation of the tangent plane to the cone at a generic point (x_0, y_0, z_0). Simplify the equation, keeping in mind that the point (x_0, y_0, z_0) satisfies the equation of the cone.
b) Show that the tangent plane always contains the origin.

4. (18pts) Let $U = \dfrac{\ln x}{xy}$, $x = \sqrt{st}$, $y = s^2 - t^2$. Use the chain rule to find $\dfrac{\partial U}{\partial s}$ when $s = 1$, $t = 2$.

5. (12pts) The range of a projectile fired at angle α with initial velocity v is given by $R = \dfrac{v^2 \sin(2\alpha)}{10}$ (R is in meters, v in meters per second, α in radians). Use differentials to estimate the change in range of a projectile fired at 40 m/s at angle $\frac{\pi}{6}$ if velocity is decreased by 0.2 meters per second, and angle is increased by 0.1 radian.

6. (12pts) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ at the point $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$, if $\tan x + \tan y + \tan z = xyz + 2$.

7. (20pts) Find and classify the local extremes for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$.

Bonus (10pts) Let $A = (0, 0)$, $B = (1, 0)$ and $C = (0, 2)$ and let d_A, d_B and d_C represent the distance from a point (x, y) to A, B and C, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2 + d_C^2$ among all points (x, y) in the triangle ABC (edges are included).

Calculus 3 — Exam 3 **MAT 309, Spring 2021 — D. Ivanšić**	**Name:** ______________________________ *Show all your work!*

1. (16pts) Let D be the region in the first quadrant bounded by the curves $y = \sqrt{x}$, $x = 0$ and $y = 2$.

a) Sketch the region D.

b) Set up $\displaystyle\iint_D \frac{1}{y^3+1}\, dA$ as iterated integrals in both orders of integration.

c) Evaluate the double integral using the easier order.

2. (12pts) Let D be the region that is under both curves $y = \sin x$ and $y = \cos x$ and above the x axis, and where $0 \leq x \leq \frac{\pi}{2}$. Set up $\iint_D x + y\, dA$, but do not evaluate the integral. Sketch the region of integration first.

3. (20pts) Use polar coordinates to find $\displaystyle\iint_D \frac{x}{x^2+y^2}\,dA$, if D is the region inside the circle $x^2+y^2=\frac{1}{4}$, and outside the cardioid $r=1+\cos\theta$. Sketch the region of integration first.

4. (18pts) Sketch the region E in the first octant $(x, y, z \geq 0)$ that is inside the cylinder $y^2+z^2=4$ and "behind" the plane $y=3x$. Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dz\,dy\,dx$ and $dy\,dz\,dx$.

5. (20pts) Use cylindrical or spherical coordinates to evaluate $\iiint_E z\,dV$, if E is the region that is above the cone $z = \sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 9$. Sketch the region E.

6. (14pts) Use cylindrical coordinates to set up the integral for the volume of a spherical cap, the region inside the sphere $x^2 + y^2 + z^2 = a^2$ that is above the plane $z = b$, where $a > 0$ and $0 \leq b \leq a$. Do not evaluate the integral. Sketch the region E.

Bonus (10pts) Sketch the surfaces given by the equations:

$$z = \frac{1}{\sqrt{x^2 + y^2}} \qquad\qquad \rho = 1 + \sin\phi$$

Calculus 3 — Exam 4
MAT 309, Spring 2021 — D. Ivanšić

Name: ______________________

Show all your work!

1. (16pts) Let $\mathbf{F}(x, y) = \langle x - y, x + y \rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative? Now, can you justify it just by looking at the picture?

2. (20pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C \dfrac{xyz}{x^2 + z^2}\, ds$, where C is the helix $x = 3t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq 2\pi$..
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x, y) = \left\langle \dfrac{x - y}{x + y + 4}, \dfrac{x + y}{x + y + 4} \right\rangle$, where C is part of the circle $x^2 + y^2 = 9$ from point $(0, 3)$ to point $(-3, 0)$, going the short way.

3. (16pts) Let $\mathbf{F}(x,y) = \langle 2x, 8y \rangle$. It is easy to see that $\mathbf{F} = \nabla f$, where $f(x,y) = x^2 + 4y^2$. Apply the fundamental theorem for line integrals to:

a) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the circle of radius 2, centered at $(1,0)$.

b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve from $(0,0)$ to $(1,2)$. (Why is the curve not specified?)

c) Sketch the directions of the vector field $\mathbf{F}$ by exploiting the function f. Very little computation is needed here.

4. (18pts) Consider the region D inside the triangle with vertices $(0,0)$, $(2,0)$ and $(2,1)$.

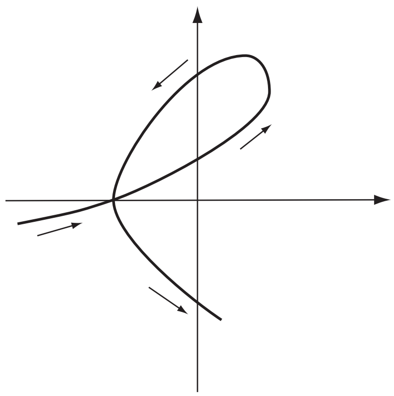
a) Draw the region.

b) Use Green's theorem to find the line integral $\int_C (y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy$, where C is the boundary of the region D, traversed counterclockwise. (Scary-looking, but it's not!)

5. (10pts) Suppose a particle moves in the velocity field $\mathbf{v}(x, y) = \langle x^2 - y^2, xy \rangle$. If it is at point $(1, 3)$ at time $t = 2$, estimate its location at time $t = 2.1$.

6. (20pts) Let $\mathbf{F}(x, y) = \left\langle \dfrac{2x}{x^2 + y}, e^y + \dfrac{1}{x^2 + y} \right\rangle$.
a) Find the domain of f: it has two parts, and consider the part that contains $(0, 1)$.
b) Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$.
c) Is $\mathbf{F}$ is conservative? Your justification should say something about the domain.
d) If the field is conservative, find its potential function.

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.



Calculus 3 — Final Exam **MAT 309, Spring 2021 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

1. (12pts) Find the equation of the plane that contains the lines given by parametric equations: $x = -2 - t$, $y = 12 + 3t$, $z = 2 + 2t$ and $x = 7 - 6t$, $y = 1 + 2t$, $z = -3 - t$. (These lines intersect — or they wouldn't determine a plane — but the point of intersection is not needed, so don't look for it.)

2. (20pts) Consider the function $f(x, y) = \dfrac{y}{x}$ on domain $\{(x, y) \mid x > 0\}$.
a) Sketch the contour map for the function, drawing level curves for levels $k = 0, \frac{1}{2}, 1, 2, -1, -\frac{1}{2}$.
b) At point $(3, -2)$, find the directional derivative of f in the direction of $\langle -1, 1\rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?
c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$.
Apply the fundamental theorem for line integrals to answer:
d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is part of the unit circle from $(0, 1)$ to $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$?
e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k = 3$ to any point on level curve $k = -2$?

3. (12pts) Find the equation of the tangent plane to the surface $x + y + z = e^{xyz}$ at point $(2, 0, -1)$.

4. (14pts) Find and classify the local extremes for $f(x, y) = x^2y + 2xy^2 + 3y$.

5. (16pts) Let D be the region bounded by the curves $y = e^x$, $y = e$ and $x = 0$.

a) Sketch the region D.

b) Set up $\displaystyle\iint_D \frac{1}{y}\, dA$ as iterated integrals in both orders of integration.

c) Evaluate the double integral using the order you find easier.

6. (18pts) Sketch the region E in the first octant ($x, y, z \geq 0$) that is inside the cylinder $y^2 + z^2 = 4$ and "behind" the plane $y = 3x$. Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dz\, dy\, dx$ and $dy\, dz\, dx$.

7. (14pts) Use either cylindrical or spherical coordinates to find the volume of a spherical cap E, the region inside the sphere $x^2 + y^2 + z^2 = 8$ that is above the plane $z = 2$. Sketch the region E.

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \dfrac{x+y}{xy+1}\, ds$, where C is the part of the curve $y = x^3 - x$ from $(-1, 0)$ to $(1, 0)$.

9. (20pts) Consider the region inside the circle $x^2 + y^2 = 4$ and above the lines $y = \sqrt{3}x$ and $y = -\frac{1}{\sqrt{3}}x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C y^3\,dx + x^3\,dy$, where C is the boundary of the region D, traversed counterclockwise.

10. (12pts) The range of a projectile fired at angle α with initial velocity v is given by $R = \dfrac{v^2 \sin(2\alpha)}{10}$ (R is in meters, v in meters per second, α in radians). Use differentials to estimate the change in range of a projectile fired at 70 m/s at angle $\frac{\pi}{3}$ if velocity is increased by 5 meters per second, and angle is decreased by 0.2 radians.

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.