Calculus 3 — Final Exam **MAT 309, Spring 2021 — D. Ivanšić**

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Show all your work!

1. (12pts) Find the equation of the plane that contains the lines given by parametric equations: $x = -2 - t$, $y = 12 + 3t$, $z = 2 + 2t$ and $x = 7 - 6t$, $y = 1 + 2t$, $z = -3 - t$. (These lines intersect — or they wouldn't determine a plane — but the point of intersection is not needed, so don't look for it.)

A point in the plane is $(-2, 12, 2)$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 2 \\ -6 & 2 & -1 \end{vmatrix} = (-3-4)\vec{i} - (1+12)\vec{j} + (-2+18)\vec{k} = -7\vec{i} - 13\vec{j} + 16\vec{k}$$

Take $\vec{n} = 7\vec{i} + 13\vec{j} - 16\vec{k}$

$$7(x-(-2)) + 13(y-12) - 16(z-2) = 0$$

$$7x + 13y - 16z + 14 - 156 + 32 = 0$$

$$7x + 13y - 16z = 110$$

2. (18pts) Consider the function $f(x, y) = \dfrac{y}{x}$ on domain $\{(x, y) \mid x > 0\}$.

a) Sketch the contour map for the function, drawing level curves for levels $k = 0, \frac{1}{2}, 1, 2, -1, -\frac{1}{2}$.

b) At point $(3, -2)$, find the directional derivative of f in the direction of $\langle -1, 1 \rangle$. In what direction is the directional derivative the greatest? What is the directional derivative in that direction?

c) Let $\mathbf{F} = \nabla f$. Sketch the vector field $\mathbf{F}$.

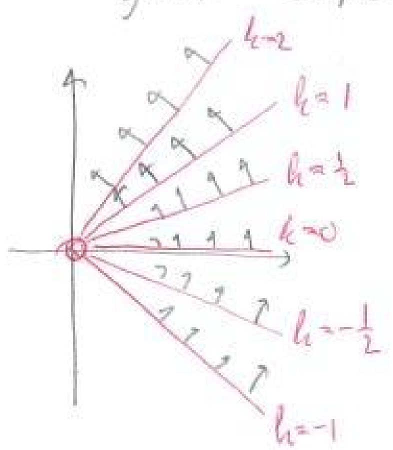
Apply the fundamental theorem for line integrals to answer:

d) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is part of the unit circle from $(\cancel{0}, \cancel{1})$ $(1, 0)$ to $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$?

e) What is $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve going from any point on level curve $k = 3$ to any point on level curve $k = -2$?

a) $\frac{y}{x} = k$ line with slope k

$y = kx$



b) Need $\nabla f = \left\langle -\frac{y}{x^2}, \frac{1}{x} \right\rangle$ $\quad \nabla f(3,-2) = \left\langle \frac{2}{9}, \frac{1}{3} \right\rangle$

$u = \dfrac{1}{\sqrt{(-1)^2 + 1^2}} = \dfrac{1}{\sqrt{2}} \langle -1, 1 \rangle$

$-\frac{2}{9} + \frac{3}{9}$

$D_u f = \nabla f \cdot \vec{u} = \frac{1}{\sqrt{2}} \langle -1, 1 \rangle \cdot \left\langle \frac{2}{9}, \frac{1}{3} \right\rangle = \frac{1}{\sqrt{2}}\left(-\frac{2}{9} + \frac{1}{3}\right) = \frac{1}{9\sqrt{2}}$

Greatest in direction $\nabla f = \left\langle \frac{2}{9}, \frac{1}{3} \right\rangle$

Max dir. derivative is $|\nabla f| = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4+9}{81}} = \frac{\sqrt{13}}{9}$

d) $\int_C \vec{F} \cdot d\vec{r} = f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) - f(1, 0) = \dfrac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} - \frac{0}{1} = -1$

e) $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = -2 - 3 = -5$

B, A pts on level curves -2, 3, respectively

3. (10pts) Find the equation of the tangent plane to the surface $x + y + z = e^{xyz}$ at point $(2, 0, -1)$.

$f(x,y,z) = x+y+z-e^{xyz}$

$\nabla f = (1-yze^{xyz}, 1-xze^{xyz}, 1-xye^{xyz})$

$\nabla f(2,0,-1) = \langle 1, 1-2\cdot(-1)\cdot 1, 1 \rangle$

$= \langle 1, 3, 1 \rangle$

Eq. of tan. plane:

$1\cdot(x-2)+3(y-0)+1\cdot(z-(-1)) = 0$

$x-2+3y+z+1=0$

$x+3y+z=1$

4. (16pts) Find and classify the local extremes for $f(x, y) = x^2y + 2xy^2 + 3y$.

$\frac{\partial f}{\partial x} = 2xy+2y^2$

$\frac{\partial f}{\partial y} = x^2+4xy+3$

$\begin{cases} 2y(x+y)=0 \\ x^2+4xy+3=0 \end{cases} \Rightarrow$ $y=0$ or $x+y=0$

$y=0$, put in 2nd eq.: $x^2+3=0$, $x^2=-3$, no real sol.

$x+y=0$, $y=-x$, put in 2nd eq.: $x^2+4x(-x)+3=0$, $-3x^2+3=0$, $x^2=1$, $x=\pm 1$

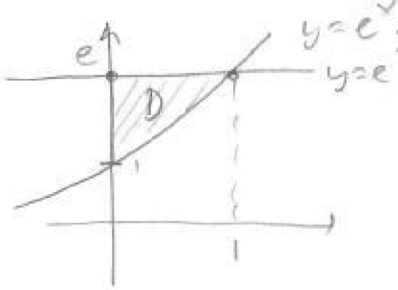
Critical points: $(-1,1), (1,-1)$

$D = \begin{vmatrix} 2y & 2x+4y \\ 2x+4y & 4x \end{vmatrix}$

$D(-1,1) = \begin{vmatrix} 2 & 2 \\ 2 & -4 \end{vmatrix} = -12 < 0$ saddle point

$D(1,-1) = \begin{vmatrix} -2 & -2 \\ -2 & 4 \end{vmatrix} = -12 < 0$ saddle point

5. (16pts) Let D be the region bounded by the curves $y = e^x$, $y = e$ and $x = 0$.
a) Sketch the region D.
b) Set up $\iint_D \frac{1}{y}\, dA$ as iterated integrals in both orders of integration.
c) Evaluate the double integral using the order you find easier.



$y = e^x,\ x = \ln y$

$$\iint_D \frac{1}{y}\, dA = \int_0^1 \int_{e^x}^{e} \frac{1}{y}\, dy\, dx = (1)$$

$$= \int_1^e \int_0^{\ln y} \frac{1}{y}\, dx\, dy = (2)$$

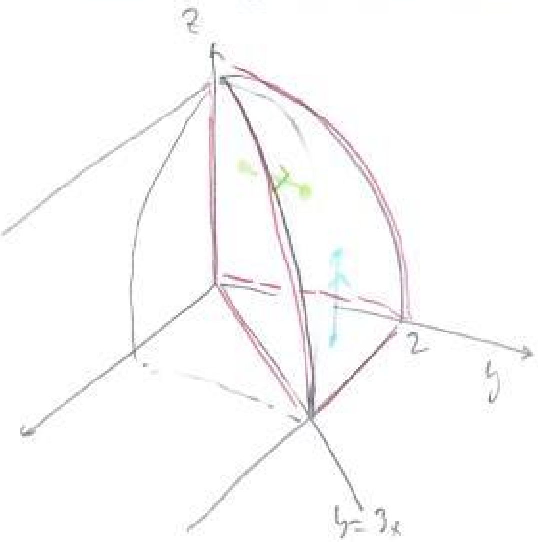
$$(1) = \int_0^1 \ln y \Big|_{e^x}^{e} dx = \int_0^1 1 - x\, dx = 1\cdot(1-0) - \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

(Neither is too hard)

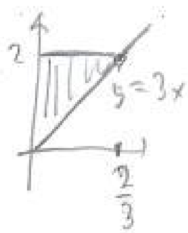
$$(2) = \int_1^e \frac{1}{y}(\ln y - 0)\, dy = \int_1^e \frac{\ln y}{y}\, dy = \frac{1}{2}(\ln y)^2 \Big|_1^e$$

$$= \frac{1}{2}\left((\ln e)^2 - \ln 1)^2\right) = \frac{1}{2}\cdot 1^2 = \frac{1}{2}$$

6. (18pts) Sketch the region E in the first octant ($x, y, z \geq 0$) that is inside the cylinder $y^2 + z^2 = 4$ and "behind" the plane $y = 3x$. Then write the two iterated triple integrals that stand for $\iiint_E f\, dV$ which end in $dz\, dy\, dx$ and $dy\, dz\, dx$.



Proj. to xy plane:



$$\int_0^{2/3} \int_{3x}^{2} \int_0^{\sqrt{4-y^2}} f\, dz\, dy\, dx$$

Proj. to xz plane

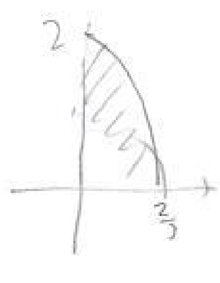
$$\int_0^{\frac{2}{3}} \int_0^{\sqrt{4-9x^2}} \int_{3x}^{\sqrt{4-z^2}} f\, dy\, dz\, dx$$

$\begin{cases} y^2 + z^2 = 4 \\ y = 3x \end{cases}$

$9x^2 + z^2 = 4$

$\frac{x^2}{\frac{4}{9}} + \frac{z^2}{4} = 1$

$z = \sqrt{4 - 9x^2}$



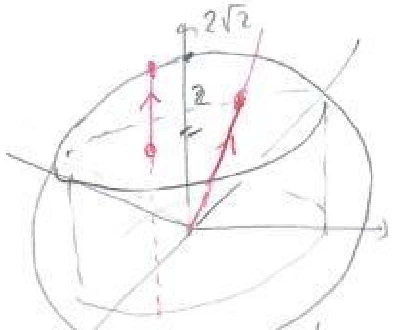
$* = 2\pi \int_0^2 r\left(\sqrt{8-r^2}-2\right)dr = 2\pi \int_0^2 r(8-r^2)^{1/2} - 2r\,dr$

$= 2\pi\left(-\frac{2}{3}\frac{(8-r^2)^{\frac{3}{2}}}{2}\Big|_0^2 - r^2\Big|_0^2\right) = 2\pi\left(-\frac{1}{3}\left(4^{\frac{3}{2}} - 8^{\frac{3}{2}}\right) - 4\right) = 2\pi\left(-\frac{20}{3} + \frac{16\sqrt{2}}{3}\right)$

$8 \qquad 8\sqrt{8} = 16\sqrt{2} \qquad = \left(\frac{32\sqrt{2}-40}{3}\right)\pi$

7. (18pts) Use either cylindrical or spherical coordinates to find the volume of a spherical cap E, the region inside the sphere $x^2+y^2+z^2=8$ that is above the plane $z=2$. Sketch the region E.

radius $= \sqrt{8} = 2\sqrt{2}$



Spherical: $\frac{z}{\rho} = \cos\phi$ $\qquad \cos\phi = \frac{1}{\sqrt{2}}$

$\frac{2}{2\sqrt{2}} = \cos\phi$ $\qquad \phi = \frac{\pi}{4}$

$z = 2$ is $\rho\cos\phi = 2$, $\rho = \frac{2}{\cos\phi}$

$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{2}{\cos\phi}}^{2\sqrt{2}} 1\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\frac{\pi}{4}}\sin\phi\,\frac{\rho^3}{3}\Big|_{\frac{2}{\cos\phi}}^{2\sqrt{2}}d\phi$

$= \frac{2\pi}{3}\int_0^{\pi/4}\sin\phi\left(\underbrace{(2\sqrt{2})^3}_{16\sqrt{2}} - \frac{8}{\cos^3\phi}\,d\phi\right) = \frac{32\sqrt{2}\pi}{3}\int_0^{\pi/4}\sin\phi\,d\phi$

$- \frac{16\pi}{3}\int_0^{\pi/4}\frac{\sin\phi}{\cos^3\phi}d\phi = \frac{32\sqrt{2}\pi}{3}(-\cos\phi)\Big|_0^{\frac{\pi}{4}} - \frac{16\pi}{3}\left(\frac{1}{2}\right)\frac{1}{\cos^2\phi}\Big|_0^{\pi/4}$

$= -\frac{32\sqrt{2}\pi}{3}\left(\frac{\sqrt{2}}{2}-1\right) - \frac{8\pi}{3}\left(\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{1^2}\right) = -\frac{32\pi}{3} + \frac{32\sqrt{2}\pi}{3} - \frac{8\pi}{3} = \left(\frac{32\sqrt{2}-40}{3}\right)\pi$

$2 - 1$

Cylindrical:

Projection is a disk:

$x^2+y^2+2^2 = 8 \to (z = \sqrt{8-r^2})$

$x^2+y^2 = 4$

$V = \int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} 1\cdot r\,dz\,dr\,d\theta = *$

8. (10pts) Set up and simplify the set-up, but do NOT evaluate the integral: $\int_C \mathbf{F}\cdot d\mathbf{r}$, if $\mathbf{F}(x,y) = \langle ye^x, \sin(xz), z^2y\rangle$, where C is the line segment from $(0,3,-1)$ to $(1,3,4)$.

Line segment is: $\vec{r}_0 + t\vec{v}$, $0 \le t \le 1$

$\vec{v} = \langle 1, 0, 5\rangle$

$x = 0 + t$

$y = 3 \qquad 0 \le t \le 1$

$z = -1 + 5t$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(t, 3, -1+5t)\cdot\vec{r}'(t)\,dt$

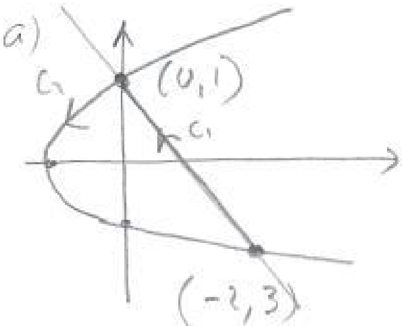
$= \int_0^1 \langle 3e^t, \sin(t(-1+5t)), (-1+5t)^2\cdot 3\rangle\cdot\langle 1, 0, 5\rangle\,dt$

$= \int_0^1 3e^t + 0 + (5t-1)^2\cdot 3\cdot 5\,dt$

$= \int_0^1 3e^t + 15(25t^2+10t+1)\,dt = \int_0^1 e^t + 375t^2 + 150t + 15\,dt$

9. (20pts) Consider the region bounded by the curves $x = y^2 - 1$ and $y = 1 - x$.
a) Draw the region.
b) Use Green's theorem to find the line integral $\int_C (y \sin x + xy \cos x)\, dx + (x \sin x + xy^2)\, dy$, where C is the boundary of the region D, traversed counterclockwise.

a)



$y^2 - 1 = 1 - y$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2, 1$

b) $\int_C (y \sin x + xy \cos x)dx + (x \sin x + xy^2)dy$

$= \iint_D \frac{\partial}{\partial x}(x \sin x + xy^2) - \frac{\partial}{\partial y}(y \sin x + xy \cos x)\, dA$

$= \iint_D 1 \cdot \cancel{\sin x} + \cancel{x \cdot \cos x} + y^2 - (\cancel{\sin x} + \cancel{x \cos x})\, dA$

$= \int_{-2}^{1} \int_{y^2-1}^{1-y} y^2\, dx\, dy = \int_{-2}^{1} y^2(1 - y - (y^2 - 1))\, dy$

$= \int_{-2}^{1} y^2(2 - y - y^2)dy = \int_{-2}^{1} 2y^2 - y^3 - y^4\, dy$

$= \left(\frac{2y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5}\right)\Big|_{-2}^{1} = \frac{2}{3}(1-(-8)) - \frac{1}{4}(1-16) - \frac{1}{5}(1-(-32))$

$= \frac{2}{3} \cdot 9 + \frac{15}{4} - \frac{33}{5} = 6 + \frac{75-132}{20} = \frac{63}{20}$

10. (12pts) The range of a projectile fired at angle α with initial velocity v is given by $R = \dfrac{v^2 \sin(2\alpha)}{10}$ (R is in meters, v in meters per second, α in radians). Use differentials to estimate the change in range of a projectile fired at 70 m/s at angle $\frac{\pi}{3}$ if velocity is increased by 5 meters per second, and angle is decreased by 0.2 radians.

$dR = \frac{\partial R}{\partial v} dv + \frac{\partial R}{\partial \alpha} d\alpha$

$= \frac{2v \sin(2\alpha)}{10} dv + \frac{v^2 \cdot \cos(2\alpha) \cdot 2}{10} d\alpha$

$= \frac{v \sin(2\alpha)}{5} dv + \frac{v^2 \cos(2\alpha)}{5} d\alpha$

[plug in $v = 70$, $\alpha = \frac{\pi}{3}$

$= \frac{70 \sin\left(\frac{2\pi}{3}\right)}{\cancel{5}} \cdot \cancel{5} + \frac{70^2 \cos\left(\frac{2\pi}{3}\right)}{5} \cdot (-0.2)$

49·4

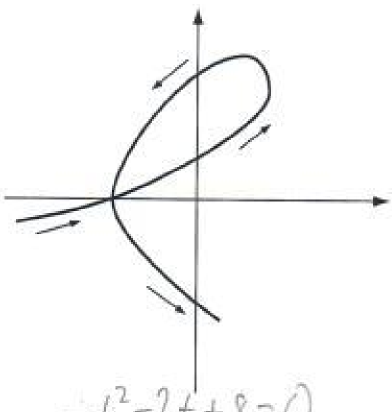
$dv = 5$

$d\alpha = -0.2$

$= 70 \cdot \frac{\sqrt{3}}{2} - \frac{\cancel{4900} \cdot \frac{1}{2}}{\cancel{5}} \cdot \frac{1}{\cancel{5}} = 35\sqrt{3} - 98$ meters

25

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.



$$\iint_D 1\,dA = \int_C x\,dy = \int_{-4}^{2} (t^3-12t)\cdot\overbrace{(-2t-2)}^{y'(t)}\,dt$$

$$= -2\int_{-4}^{2} (t^3-12t)(t+1)\,dt = -2\int_{-4}^{2} t^4+t^3-12t^2+12t\,dt$$

$$= -2\left(\frac{t^5}{5}+\frac{t^4}{4}-4t^3+6t^2\right)\Bigg|_{-4}^{2}$$

$$= -2\left(\frac{1}{5}\left(32-(-1024)\right)+\frac{1}{4}\left(16-256\right)-4\left(8-(-64)\right)+6(4-16)\right)$$

$$= -2\left(\frac{1056}{5}-\frac{240}{4}-4\cdot 72-72\right)$$

$$= -2\left(\frac{1056}{5}-60-\underbrace{5\cdot 72}_{360}\right) = -2\left(\frac{1056}{5}-420\right)$$

$$= -2\,\frac{1056-2100}{5} = 2\cdot\frac{1044}{5} = \frac{2088}{5}$$

$-t^2-2t+8=0$

$t^2+2t-8=0$

$(t-2)(t+4)=0$

$t=-4,2$

$(-4)^3=64$

$(-4)^4=256$

$(-4)^5=-1024$