Calculus 3 — Exam 4
MAT 309, Spring 2021 — D. Ivanšić

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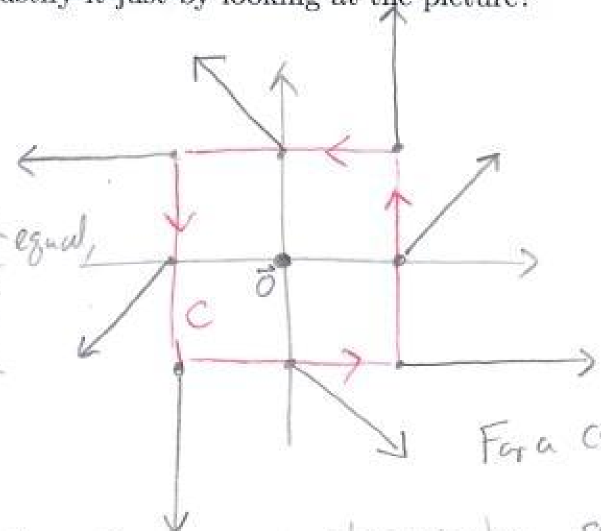
1. (16pts) Let $\mathbf{F}(x,y) = \langle x-y, x+y \rangle$.
a) Sketch the vector field by evaluating it at 9 points (for example, a 3×3 grid).
b) Is F conservative? Now, can you justify it just by looking at the picture?

a)

(x,y)	$\vec{F}(x,y)$
$(-1,-1)$	$(0,-2)$
$(0,-1)$	$(1,-1)$
$(1,-1)$	$(2,0)$
$(-1,0)$	$(-1,-1)$
$(0,0)$	$(0,0)$
$(1,0)$	$(1,1)$
$(-1,1)$	$(-2,0)$
$(0,1)$	$(-1,1)$
$(1,1)$	$(0,2)$

b) $\underbrace{\frac{\partial Q}{\partial x} = 1 \quad \frac{\partial P}{\partial y} = -1}$

These are not equal, so $\vec{F}$ is not conservative



For a conservative field it would be 0.

Integrating along C gives a positive number, since $\vec{F}$ and $\vec{r}\,'$ have angle $< \frac{\pi}{2}$ at all pts of C

2. (20pts) In both cases set up and simplify the set-up, but do NOT evaluate the integral.
a) $\int_C \frac{xyz}{x^2+z^2}\,ds$, where C is the helix $x = 3t$, $y = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$..
b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F}(x,y) = \left\langle \frac{x-y}{x+y+4}, \frac{x+y}{x+y+4} \right\rangle$, where C is part of the circle $x^2+y^2 = 9$ from point $(0,3)$ to point $(-3,0)$, going the short way.

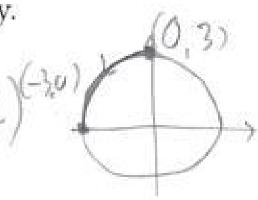
a) $\vec{r}\,'(t) = \langle 3, -\sin t, \cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{9 + (-\sin t)^2 + \cos^2 t}$

$= \sqrt{10}$

$\int \frac{xyz}{x^2+z^2}\,ds = \int_0^{2\pi} \frac{3t\cos t \sin t}{9t^2 + \sin^2 t}\sqrt{10}\,dt$

b)



$x = 3\cos t$ $\quad \frac{\pi}{2} \le t \le \pi$
$y = 3\sin t$

$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi/2}^{\pi} \left\langle \frac{3\cos t - 3\sin t}{3\cos t + 3\sin t + 4}, \frac{3\cos t + 3\sin t}{3\cos t + 3\sin t + 4} \right\rangle \cdot \langle -3\sin t, 3\cos t \rangle\,dt$

$= \int_{\pi/2}^{\pi} \frac{-9\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t}{3\cos t + 3\sin t + 4}\,dt$

$= \int_{\pi/2}^{\pi} \frac{9}{3\cos t + 3\sin t + 4}\,dt$

3. (16pts) Let $\mathbf{F}(x,y) = \langle 2x, 8y \rangle$. It is easy to see that $\mathbf{F} = \nabla f$, where $f(x,y) = x^2 + 4y^2$. Apply the fundamental theorem for line integrals to:

a) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the circle of radius 2, centered at $(1,0)$.

b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is a curve from $(0,0)$ to $(1,2)$. (Why is the curve not specified?)

c) Sketch the directions of the vector field $\mathbf{F}$ by exploiting the function f.

a) $C \cdot \vec{r}(t)$, $a \le t \le b$, $\vec{r}(a) = \vec{r}(b) = (3,0)$

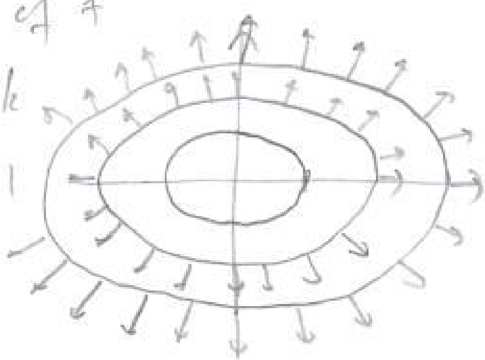
c) $\vec{F}$ is perp. to level curves of f

$$\int_C \nabla f \cdot d\vec{r} = f(3,0) - f(3,0) = 0$$

$x^2 + 4y^2 = k$

$\frac{x^2}{k} + \frac{y^2}{\frac{k}{4}} = 1$

ellipse, semiaxes $\sqrt{k}$, $\frac{\sqrt{k}}{2}$



b) $\int_C \nabla f \cdot d\vec{r} = f(1,2) - f(0,0)$

$= 1^2 + 4 \cdot 2^2 - 0$

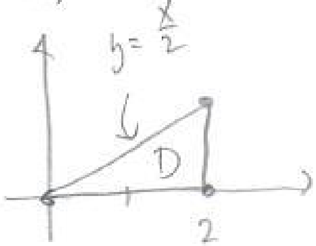
$= 17$

4. (18pts) Consider the region D inside the triangle with vertices $(0,0)$, $(2,0)$ and $(2,1)$.

a) Draw the region.

b) Use Green's theorem to find the line integral $\int_C (y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy$, where C is the boundary of the region D, traversed counterclockwise. (Scary-looking, but it's not!)

a)



$$\int_C (y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy$$

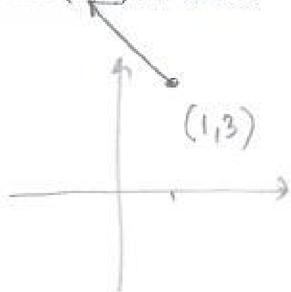
$$= \iint_D \frac{\partial}{\partial x}(xy + x\cos x) - \frac{\partial}{\partial y}(y\cos x - xy\sin x)\,dA$$

$$= \iint_D (y + \cos x - x\sin x) - (\cos x - x\sin x)\,dA$$

$$= \iint_D y\,dA = \int_0^2 \int_0^{\frac{x}{2}} y\,dy\,dx = \int_0^2 \frac{y^2}{2}\Big|_0^{\frac{x}{2}} dx$$

$$= \int_0^2 \frac{1}{2}\frac{x^2}{4}\,dx = \frac{1}{8}\frac{x^3}{3}\Big|_0^2 = \frac{1}{24}(8-0) = \frac{1}{3}$$

5. (10pts) Suppose a particle moves in the velocity field $\mathbf{v}(x, y) = \langle x^2 - y^2, xy\rangle$. If it is at point $(1, 3)$ at time $t = 2$, estimate its location at time $t = 2.1$.



$\vec{v}(1,3) = \langle 1-9, 3\rangle = \langle -8, 3\rangle$

$\vec{r}(2.1) \approx \vec{r}(2) + \Delta t \cdot \vec{v}(1,3)$

$= \langle 1,3\rangle + 0.1\langle -8, 3\rangle$

$= \langle 1,3\rangle + \langle -0.8, 0.3\rangle = (0.2, 3.3)$

Approx at point $(0.2, 3.3)$

6. (20pts) Let $\mathbf{F}(x, y) = \left\langle \dfrac{2x}{x^2+y}, e^y + \dfrac{1}{x^2+y} \right\rangle$.

a) Find the domain of f: it has two parts, and consider the part that contains $(0, 1)$.

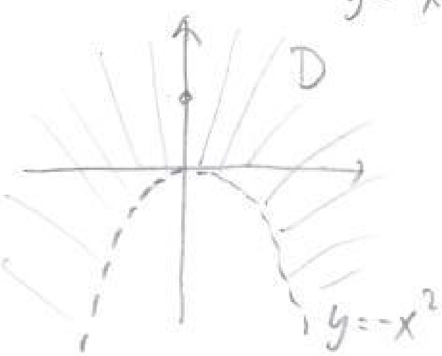
b) Compute $\dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial P}{\partial y}$.

c) Is $\mathbf{F}$ is conservative? Your justification should say something about the domain.

d) If the field is conservative, find its potential function.

a) Can't have $x^2+y=0$

$y=-x^2$



b) $\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}\left(e^y + \dfrac{1}{x^2+y}\right) = 0 - \dfrac{1}{(x^2+y)^2}\cdot 2x = -\dfrac{2x}{(x^2+y)^2}$

$\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}\dfrac{2x}{x^2+y} = -\dfrac{1}{(x^2+y)^2}\cdot 2x = -\dfrac{2x}{(x^2+y)^2}$

$\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

c) Since the domain is simply-connected and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, the field is conservative

d) $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y}$

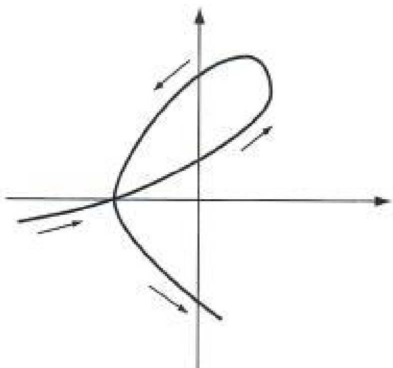
$f(x,y) = \ln(x^2+y) + g(y)$

$e^y + \cancel{\dfrac{1}{x^2+y}} = \dfrac{\partial f}{\partial y} = \cancel{\dfrac{1}{x^2+y}} + g'(y)$

$g'(y) = e^y$, so $g(y) = e^y + C$

$f(x,y) = \ln(x^2+y) + e^y + C$

Bonus. (10pts) Pictured is a spring 2020 friend from calculus 2, the curve parametrized by $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$. Use Green's theorem to find the area of the loop.



$$\iint_D 1\, dA = \int_C x\, dy = \int_{-4}^{2} (t^3 - 12t) \cdot \underbrace{(-2t-2)}_{y'(t)}\, dt$$

$$= -2\int_{-4}^{2} (t^3 - 12t)(t+1)\, dt = -2\int_{-4}^{2} t^4 + t^3 - 12t^2 + 12t\, dt$$

$$= -2\left(\frac{t^5}{5} + \frac{t^4}{4} - 4t^3 + 6t^2\right)\Big|_{-4}^{2}$$

$$= -2\left(\frac{1}{5}\left(32 - (-1024)\right) + \frac{1}{4}\left(16 - 256\right) - 4\left(8 - (-64)\right) + 6(4 - 16)\right)$$

$$= -2\left(\frac{1056}{5} - \frac{240}{4} - 4 \cdot 72 - 72\right)$$

$$= -2\left(\frac{1056}{5} - 60 - \underbrace{5 \cdot 72}_{360}\right) = -2\left(\frac{1056}{5} - 420\right)$$

$$= -2\,\frac{1056 - 2100}{5} = 2 \cdot \frac{1044}{5} = \frac{2088}{5}$$

$-t^2 - 2t + 8 = 0$

$t^2 + 2t - 8 = 0$

$(t-2)(t+4) = 0$

$t = -4, 2$

$(-4)^3 = 64$

$(-4)^4 = 256$

$(-4)^5 = -1024$