Calculus 3 — Exam 3
MAT 309, Spring 2021 — D. Ivanšić

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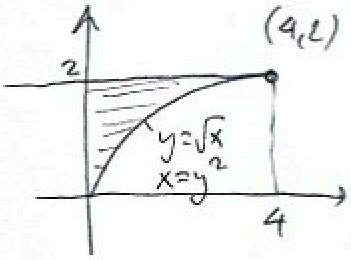
Show all your work!

1. (16pts) Let D be the region in the first quadrant bounded by the curves $y=\sqrt{x}$, $x=0$ and $y=2$.

a) Sketch the region D.

b) Set up $\iint_D \frac{1}{y^3+1}\,dA$ as iterated integrals in both orders of integration.

c) Evaluate the double integral using the easier order.

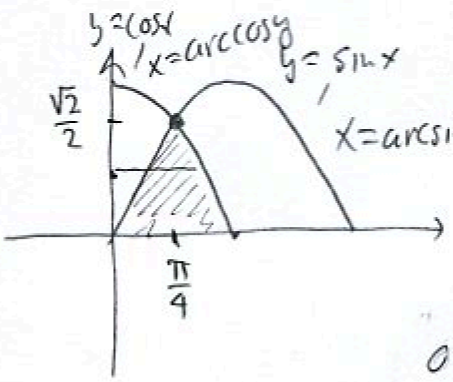


$$\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx$$

$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy = \int_0^2 \frac{1}{y^3+1}(y^2-0)\,dy$$

$$= \int_0^2 \frac{y^2}{y^3+1}\,dy = \frac{1}{3}\ln|y^3+1|\Big|_0^2 = \frac{1}{3}(\ln 9 - \ln 1) = \frac{\ln 9}{3}$$

2. (12pts) Let D be the region that is under both curves $y=\sin x$ and $y=\cos x$ and above the x axis, and where $0\le x\le \frac{\pi}{2}$. Set up $\iint_D x+y\,dA$, but do not evaluate the integral. Sketch the region of integration first.



$$\int_0^{\frac{\sqrt{2}}{2}} \int_{\arcsin y}^{\arccos y} x+y\,dx\,dy$$

OR: $\displaystyle\int_0^{\pi/4}\int_0^{\sin x} x+y\,dy\,dx + \int_{\pi/4}^{\pi/2}\int_0^{\cos x} x+y\,dy\,dx$

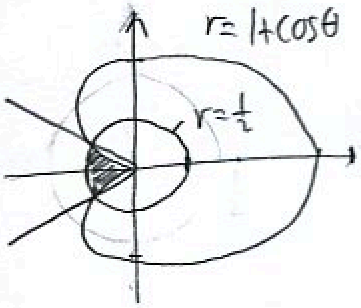
$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}$

3. (20pts) Use polar coordinates to find $\iint_D \frac{x}{x^2+y^2}\,dA$, if D is the region inside the circle $x^2+y^2=\frac{1}{4}$, and outside the cardioid $r = 1+\cos\theta$. Sketch the region of integration first.

radius $= \frac{1}{2}$



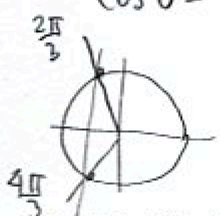
$$\iint \frac{x}{x^2+y^2}\,dA = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_{1+\cos\theta}^{\frac{1}{2}} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \cos\theta\left(\frac{1}{2}-(1+\cos\theta)\right)d\theta = \int_{2\pi/3}^{4\pi/3} -\frac{1}{2}\cos\theta - \cos^2\theta\,d\theta$$

$1+\cos\theta = \frac{1}{2}$

$\cos\theta = -\frac{1}{2}$

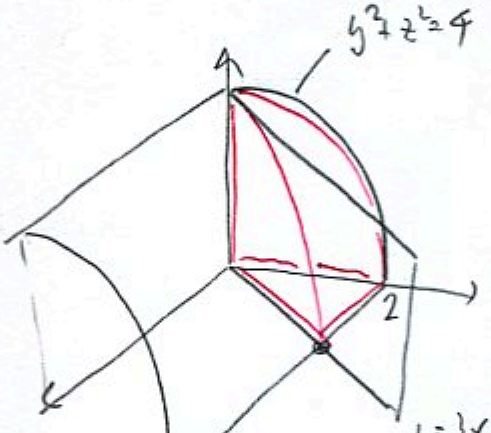
$$= \int_{2\pi/3}^{4\pi/3} -\frac{1}{2}\cos\theta - \frac{1+\cos(2\theta)}{2}\,d\theta = -\frac{1}{2}\int_{2\pi/3}^{4\pi/3} \cos\theta + 1 + \cos(2\theta)\,d\theta$$



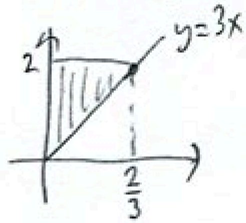
$$= -\frac{1}{2}\left(\sin\theta\Big|_{2\pi/3}^{4\pi/3} + \left(\frac{4\pi}{3}-\frac{2\pi}{3}\right) + \frac{\sin 2\theta}{2}\Big|_{2\pi/3}^{4\pi/3}\right) = -\frac{1}{2}\left(\underbrace{-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}}_{-\sqrt{3}} + \frac{2\pi}{3} + \underbrace{\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\left(-\frac{\sqrt{3}}{2}\right)\right)}_{+\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{4} - \frac{\pi}{3}$$

4. (18pts) Sketch the region E in the first octant $(x, y, z \geq 0)$ that is inside the cylinder $y^2+z^2=4$ and "behind" the plane $y = 3x$. Then write the two iterated triple integrals that stand for $\iiint_E f\,dV$ which end in $dz\,dy\,dx$ and $dy\,dz\,dx$.

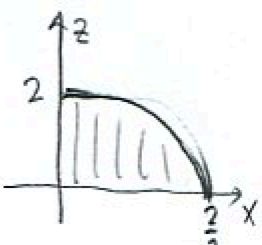


Proj. to xy:



$$\int_0^{\frac{2}{3}} \int_{3x}^{2} \int_0^{\sqrt{4-y^2}} f\,dz\,dy\,dx$$

Proj. to xz



$$\int_0^{\frac{2}{3}} \int_0^{\sqrt{4-9x^2}} \int_{3x}^{\sqrt{4-z^2}} f\,dy\,dz\,dx$$

$y^2+z^2=4$

$y=3x$

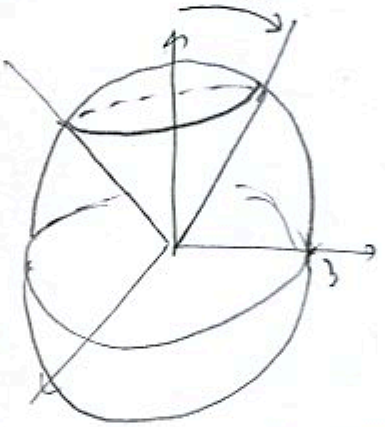
$\frac{x^2}{4/9}+\frac{z^2}{4}=1$

$(3x)^2+z^2=4$

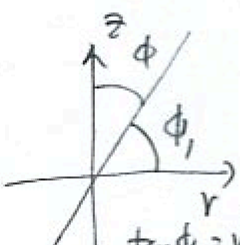
$z = \pm\sqrt{4-9x^2}$

5. (20pts) Use cylindrical or spherical coordinates to evaluate $\iiint_E z\,dV$, if E is the region that is above the cone $z = \sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 9$. Sketch the region E.

$\pi/6$



$z = \sqrt{3r^2} = \sqrt{3}\,r$



$\tan\phi_1 = \sqrt{3}$

$\phi_1 = \frac{\pi}{3}$

$\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

radius = 3

$\boxed{\text{Spherical}}$

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\int_0^{3} \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

no θ

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{6}} \cos\phi\sin\phi \cdot \int_0^{3} \rho^3\,d\rho$$

$$= 2\pi \cdot \left.\frac{\sin^2\phi}{2}\right|_0^{\frac{\pi}{6}} \cdot \left.\frac{\rho^4}{4}\right|_0^{3}$$

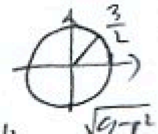
$$= \cancel{2}\pi \cdot \frac{1}{\cancel{2}}\left(\left(\tfrac{1}{2}\right)^2 - 0\right)\cdot\frac{1}{4}\left(3^4 - 0\right) = \frac{81}{16}\pi$$

Cylindrical:

$\left.\begin{array}{l} z = \sqrt{3}\,r \\ r^2 + z^2 = 9 \end{array}\right\} \Rightarrow r^2 + 3r^2 = 9$

$4r^2 = 9,\quad r = \frac{3}{2}$

proj to xy

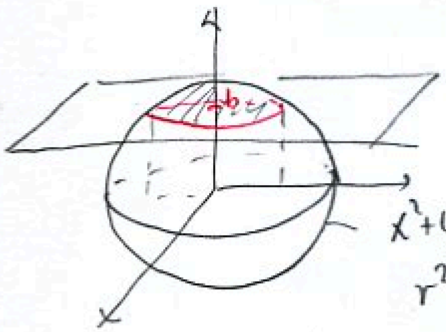


$$\int_0^{2\pi}\int_0^{3/2}\int_{\sqrt{3}r}^{\sqrt{9-r^2}} z r\,dz\,dr\,d\theta = \cancel{2}\pi \cdot \int_0^{3/2} r \left.\frac{z^2}{\cancel{2}}\right|_{\sqrt{3}r}^{\sqrt{9-r^2}} dr$$

$$= \pi\int_0^{3/2} r\left(\sqrt{9-r^2}^{\,2} - (\sqrt{3}r)^2\right)dr = \pi\int_0^{3/2} r\left(9 - r^2 - 3r^2\right)dr = \pi\int_0^{3/2} 9r - 4r^3\,dr = \pi\left.\left(9\frac{r^2}{2} - r^4\right)\right|_0^{3/2} = \pi\left(\frac{81}{8} - \frac{81}{16}\right)$$

$$= \frac{81}{16}\pi$$

6. (14pts) Use cylindrical coordinates to set up the integral for the volume of a spherical cap, the region inside the sphere $x^2 + y^2 + z^2 = a^2$ that is above the plane $z = b$, where $a > 0$ and $0 \le b \le a$. Do not evaluate the integral. Sketch the region E.



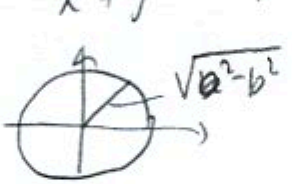
$x^2 + y^2 + z^2 = a^2$

$r^2 + z^2 = a^2$

$$\int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}}\int_b^{\sqrt{a^2-r^2}} 1 \cdot r\,dz\,dr\,d\theta$$

$\left.\begin{array}{l} z = b \\ x^2 + y^2 + z^2 = a^2 \end{array}\right\}\quad \begin{array}{l} x^2 + y^2 + b^2 = a^2 \\ x^2 + y^2 = a^2 - b^2 \end{array}$

proj to xy:



Bonus (10pts) Sketch the surfaces given by the equations:

$$z = \frac{1}{\sqrt{x^2 + y^2}}$$

$$\rho = 1 + \sin\phi$$