Calculus 3 — Exam 2	**Name:** Saul Ocean
MAT 309, Spring 2021 — D. Ivanšić	*Show all your work!*

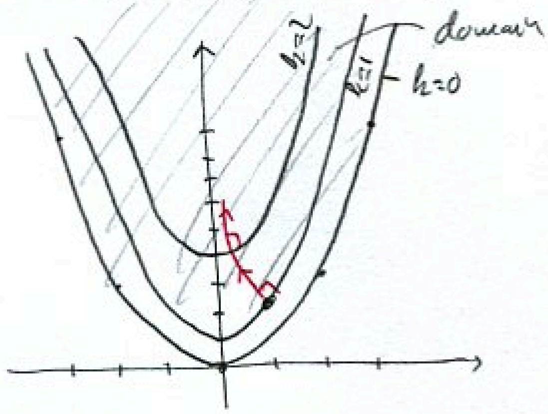
1. (12pts) Let $f(x,y) = \sqrt{y - x^2}$.
a) Find the domain of f.
b) Sketch the contour map for the function, drawing level curves for levels $k = -1, 0, 1, 2$. Note the domain on the picture.
c) Suppose $f(x,y)$ is the temperature at point (x,y) and a heat-seeking insect (always moves in direction of greatest heat increase) starts at point $(1,2)$. Sketch the path the insect will take and explain.

a) Must have:

$y - x^2 \geq 0$

$y \geq x^2$

above parabola $y = x^2$



b) $\sqrt{y-x^2} = k$

If $k<0$ nothing

$y - x^2 = k^2$

$y = x^2 + k^2$ parabolas

c) path is perpendicular to level curves

2. (16pts) Let $f(x,y) = xe^{x^3+y^3}$.
a) At point $(1,0)$, find the directional derivative of f in the direction of $\langle -2, 1 \rangle$.
b) In what direction is the directional derivative the greatest, and what is its value?

a) $\nabla f = \left\langle e^{x^3+y^3} + xe^{x^3+y^3} \cdot 3x^2,\ xe^{x^3+y^3} 3y^2 \right\rangle = \left\langle e^{x^3+y^3}(1+3x^3),\ 3xy^2 e^{x^3+y^3} \right\rangle$

$\nabla f(1,0) = e^{1+0} \langle 1+3, 0 \rangle = e \langle 4, 0 \rangle$

$\vec{u} = \frac{1}{\sqrt{(-2)^2+1^2}} \langle -2, 1 \rangle = \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$

$\nabla f(1,0) \cdot \vec{u} = e \cdot \langle 4, 0 \rangle \cdot \frac{1}{\sqrt{5}} \langle -2, 1 \rangle = \frac{e}{\sqrt{5}} (4 \cdot (-2) + 0 \cdot 1) = -\frac{8e}{\sqrt{5}}$

b) In direction of $\nabla f(1,0) = e\langle 4, 0 \rangle$

value is $|\nabla f(1,0)| = e\,|\langle 4, 0 \rangle| = 4e$

3. (12pts) Consider the elliptical cone $y^2 + 3z^2 - x^2 = 0$.
a) Find the equation of the tangent plane to the cone at a generic point (x_0, y_0, z_0). Simplify the equation, keeping in mind that the point (x_0, y_0, z_0) satisfies the equation of the cone.
b) Show that the tangent plane always contains the origin.

a) $F(x,y,z) = y^2 + 3z^2 - x^2$

$\nabla F = \langle -2x, 2y, 6z \rangle$

$\nabla F(x_0, y_0, z_0) = \langle -2x_0, 2y_0, 6z_0 \rangle$

May take $\vec{n} = (-x_0, y_0, 3z_0)$

$-x_0(x - x_0) + y_0(y - y_0) + 3z_0(z - z_0) = 0$

$-x_0x + y_0y + 3z_0z + \underbrace{x_0^2 - y_0^2 - 3z_0^2}_{=0 \text{ since } (x_0, y_0, z_0) \text{ satisfies equation}} = 0$

$\boxed{-x_0x + y_0y + 3z_0z = 0 \quad \text{eq. of tangent plane}}$

b) $\underset{\to (x,y,z)}{(0,0,0)}$ satisfies the equation, so origin is on tangent plane.

4. (16pts) Let $U = \dfrac{\ln x}{xy}$, $x = \sqrt{st}$, $y = s^2 - t^2$. Use the chain rule to find $\dfrac{\partial U}{\partial s}$ when $s = 1$, $t = 2$.

$\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y} \cdot \dfrac{\partial y}{\partial s}$

$= \dfrac{\frac{1}{x} \cdot xy - \ln x \cdot y}{(xy)^2} \cdot \dfrac{\sqrt{t}}{2\sqrt{s}} + \left(-\dfrac{\ln x}{xy^2}\right) \cdot 2s = \left[\begin{array}{ll} \text{plug in:} & \\ s = 1 & x = \sqrt{1 \cdot 2} = \sqrt{2} \\ t = 2 & y = 1^2 - 2^2 = -3 \end{array}\right.$

$= \dfrac{y(1 - \ln x)}{x^2 y} \cdot \dfrac{\sqrt{t}}{2\sqrt{s}} - \dfrac{\ln x}{xy^2} \cdot 2s$

$= \dfrac{1 - \ln\sqrt{2}}{2 \cdot (-3)} \cdot \dfrac{\sqrt{2}}{2} - \dfrac{\ln\sqrt{2}}{\cancel{\sqrt{2}} \cdot 9} \cdot \overset{\sqrt{2}}{\cancel{2}} = \dfrac{\ln\sqrt{2} - 1}{12} \cdot \sqrt{2} - \dfrac{\ln\sqrt{2}}{9}\sqrt{2} = \sqrt{2}\left(\dfrac{3\ln\sqrt{2} - 3 - 4\ln\sqrt{2}}{36}\right)$

$= -\dfrac{\ln\sqrt{2} + 3}{36}\sqrt{2}$

5. (12pts) The range of a projectile fired at angle α with initial velocity v is given by $R = \dfrac{v^2\sin(2\alpha)}{10}$ (R is in meters, v in meters per second, α in radians). Use differentials to estimate the change in range of a projectile fired at 40 m/s at angle $\frac{\pi}{6}$ if velocity is decreased by 0.2 meters per second, and angle is increased by 0.1 radian.

$$dR = \frac{\partial R}{\partial v}\,dv + \frac{\partial R}{\partial \alpha}\,d\alpha$$

$$= \frac{2v\sin(2\alpha)}{10}\,dv + \frac{v^2\cos(2\alpha)\cdot 2}{10}\,d\alpha$$

$$= \frac{v\sin(2\alpha)}{5}\,dv + \frac{v^2\cos(2\alpha)}{5}\,d\alpha = \Big[\quad \text{put in } v=40 \quad dv=-0.2, \quad \alpha=\frac{\pi}{6} \quad d\alpha=0.1$$

$$dR = \frac{40\cdot\frac{\sqrt{3}}{2}}{5}\cdot(-0.2) + \frac{40^2\frac{1}{2}}{5}\cdot 0.1 = -4\sqrt{3}\cdot 0.2 + 160\cdot 0.1$$

$$= 16 - 0.8\sqrt{3}$$

6. (12pts) Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ at the point $(0, \frac{\pi}{4}, \frac{\pi}{4})$, if $\tan x + \tan y + \tan z = xyz + 2$.

Let $F(x,y,z) = \tan x + \tan y + \tan z - xyz$

$$\frac{\partial F}{\partial x} = \sec^2 x - yz \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{\sec^2 x - yz}{\sec^2 z - xy}$$

$$\frac{\partial F}{\partial z} = \sec^2 z - xy$$

$$= -\frac{1-\frac{\pi^2}{16}}{2-0} = -\frac{16-\pi^2}{32} = \frac{\pi^2-16}{32}$$

7. (20pts) Find and classify the local extremes for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2$.

$f_x = 6xy - 6x$

$f_y = 3x^2 + 3y^2 - 6y$

$\begin{cases} 6xy - 6x = 0 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases} \rightsquigarrow \begin{cases} 6x(y-1) = 0 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases}$

From 1st eq.,

$x = 0$ or $y = 1$

$\Rightarrow 3y^2 - 6y = 0 \qquad 3x^2 - 3 = 0$

$3y(y-2) = 0 \qquad 3(x^2-1) = 0$

$y = 0, 2 \qquad x = \pm 1$

$D = \begin{vmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{vmatrix}$

(x, y)	
$(0,0)$	$\begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} > 0$ $f_{xx} < 0$, loc. max
$(0,2)$	$\begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} > 0$ $f_{xx} > 0$ loc. min
$(-1,1)$	$\begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} < 0$ saddle point,
$(1,1)$	$\begin{vmatrix} 0 & 6 \\ 0 & 0 \end{vmatrix} < 0$

Bonus (10pts) Let $A = (0, 0)$, $B = (1, 0)$ and $C = (0, 2)$ and let d_A, d_B and d_C represent the distance from a point (x, y) to A, B and C, respectively. Find the absolute maximum and minimum of $d_A^2 + d_B^2 + d_C^2$ among all points (x, y) in the triangle ABC (edges are included).

$d_A^2 + d_B^2 + d_C^2 = (x-0)^2 + (y-0)^2 + (x-1)^2 + (y-0)^2 + (x-0)^2 + (y-2)^2$

$= x^2 + y^2 + x^2 - 2x + 1 + y^2 + x^2 + y^2 - 4y + 4 = 3x^2 + 3y^2 - 2x - 4y + 5 = f(x,y)$

$f_x = 6x - 2$

$f_y = 6y - 4$

$\begin{cases} 6x - 2 = 0 \\ 6y - 4 = 0 \end{cases}$ $\boxed{x = \frac{1}{3}, \ y = \frac{2}{3}}$

(A) $x = t, y = 0 \quad 0 \le t \le 1$

$f(t, 0) = 3t^2 - 2t + 5 = g(t)$

$g'(t) = 6t - 2 \qquad 6t - 2 = 0 \Rightarrow t = \frac{1}{3}$ $\boxed{(\frac{1}{3}, 0)}$

(B) Line is $y = 2 - 2x$, $x = t, y = 2 - 2t$

$f(t, 2-2t) = 3t^2 + 3(2-2t)^2 - 2t - 8 + 8t + 5 = g(t)$

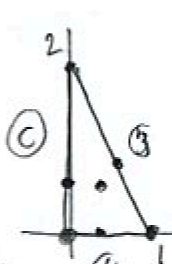
$g'(t) = 6t + 6(2-2t)(-2) - 2 + 8$

$10t - 6 = 0$, $t = \frac{3}{5}$ $\boxed{(\frac{3}{5}, \frac{4}{5})}$

(C) $x = 0, y = t \quad 0 \le t \le 2$

$f(0, t) = 3t^2 - 4t + 5 = g(t)$

$g'(t) = 6t - 4$, $t = \frac{2}{3}$ $\boxed{(0, \frac{2}{3})}$



Evaluate (x, y)	$f(x,y)$	
$(\frac{1}{3}, \frac{2}{3})$	$\frac{1}{3} + \frac{4}{3} - \frac{2}{3} - \frac{8}{3} = -\frac{5}{3} + 5$	min
$(\frac{1}{3}, 0)$	$\frac{1}{3} - \frac{2}{3} = -\frac{1}{3} + 5$	
$(\frac{3}{5}, \frac{4}{5})$	$\frac{27}{25} + \frac{48}{25} - \frac{6}{5} - \frac{16}{5} = -\frac{35}{25} = -\frac{7}{5} + 5$	
$(0, \frac{2}{3})$	$\frac{4}{3} - \frac{8}{3} = -\frac{4}{3} + 5$	

x, y	
$(0,0)$	$0 + 5$
$(1,0)$	$1 + 5$
$(0,2)$	$4 + 5$ max