Calculus 3 — Exam 1
MAT 309, Spring 2021 — D. Ivanšić

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1. (11pts) Let $\mathbf{u} = \langle 1, 3, -1 \rangle$ and $\mathbf{v} = \langle 0, 2, 1 \rangle$.
a) Calculate $-2\mathbf{u}$, $3\mathbf{v} - 4\mathbf{u}$, and $\mathbf{u} \cdot \mathbf{v}$.
b) Find a vector of length $\sqrt{5}$ in direction of $\mathbf{u}$.
c) If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, find $\cos\theta$.

a) $-2\vec{u} = \langle -2, -6, 2 \rangle$

$3\vec{v} - 4\vec{u} = \langle 0, 6, 3 \rangle - \langle 4, 12, -4 \rangle$

$= \langle -4, -6, 7 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 0 + 3 \cdot 2 + (-1) \cdot 1 = 5$

b) $|\vec{u}| = \sqrt{1^2 + 3^2 + (-1)^2} = \sqrt{11}$

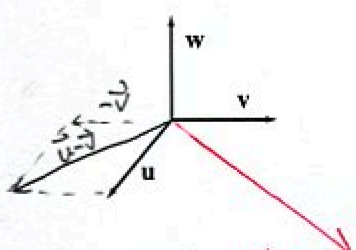
$\frac{\sqrt{5}}{\sqrt{11}} \langle 1, 3, -1 \rangle$ has length $\sqrt{5}$

c) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$

$= \frac{5}{\sqrt{11}\sqrt{0^2+2^2+1^2}} = \frac{5}{\sqrt{11}\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{11}}$

$\cos\theta = \sqrt{\frac{5}{11}}$

2. (12pts) In the picture, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually perpendicular and all have length 3.
a) Draw the vector $\mathbf{u} - \mathbf{v}$ with its tail coinciding with the other tails.
b) Which is longer (if any): $\mathbf{u} \times \mathbf{v}$ or $\mathbf{u} \times (\mathbf{u} - \mathbf{v})$?
c) Draw the the vector $\mathbf{w} \times (\mathbf{u} - \mathbf{v})$. Accurate length is not important.



a, c) $\vec{w} \times (\vec{u} - \vec{v})$

b) Since $\vec{u} \times (\vec{u} - \vec{v}) = \vec{u} \times \vec{u} - \vec{u} \times \vec{v} = -\vec{u} \times \vec{v}$ and $\vec{u} \times \vec{v}$ and $-\vec{u} \times \vec{v}$ have same length, they have same length

Or, computing lengths:

$|\vec{u} \times \vec{v}| = 3 \cdot 3 \cdot \sin\frac{\pi}{2} = 9$

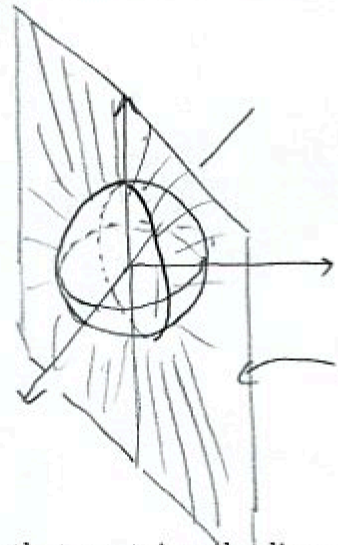
$|\vec{u} \times (\vec{u} - \vec{v})| = 3 \cdot 3\sqrt{2} \cdot \sin\frac{\pi}{4} = 3 \cdot 3\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 9$

3. (8pts) Draw the set in $\mathbf{R}^3$ described by:

$x^2 + y^2 + z^2 \geq 1$, $y = x$

$x^2 + y^2 + z^2 \geq 1$ ↓ outside of sphere centered at O, radius 1

$y = x$ ↓ plane, perp to xy-plane goes through line $y = x$



← region of plane, outside of sphere.

4. (12pts) Find the equation of the plane that contains the lines given by parametric equations: $x = 1 + 2t$, $y = -2 - t$, $z = -3 + 4t$ and $x = 5 - t$, $y = -4 + 3t$, $z = 5 + t$. (These lines intersect — or they wouldn't determine a plane — but the point of intersection is not needed, so don't look for it.)

line 1: pt. $(1,-2,-3)$, $\vec{v}_1 = \langle 2,-1,4 \rangle$

line 2 pt $(5,-4,5)$, $\vec{v}_2 = \langle -1, 3, 1 \rangle$

Pt. in plane: $(1,-2,-3)$

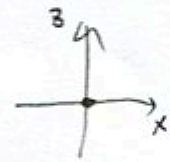
$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ -1 & 3 & 1 \end{vmatrix} = (-1-12)\vec{i} - (2+4)\vec{j} + (6-1)\vec{k} = -13\vec{i} - 6\vec{j} + 5\vec{k}$$

Take $\vec{n} = \langle 13, 6, -5 \rangle$

$13(x-1) + 6(y+2) - 5(z+3) = 0$

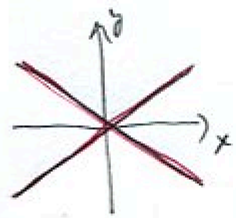
$13x + 6y + 5z = 16$

5. (16pts) This problem is about the surface $x^2 - 2y^2 + 5z^2 = 0$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.



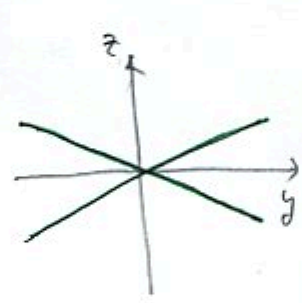
a) Traces in:

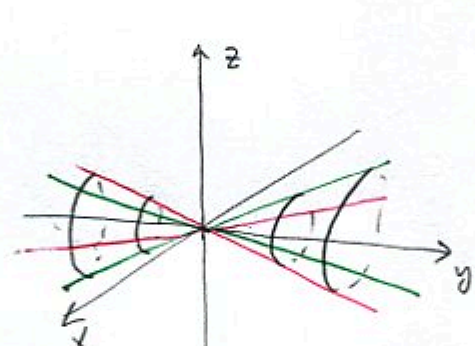
xy-plane $z=0$: $x^2 - 2y^2 = 0$, $y^2 = \frac{x^2}{2}$, $y = \pm\frac{x}{\sqrt{2}}$ two lines

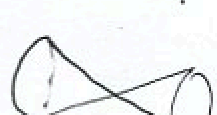


xz-plane $y=0$: $x^2 + 5z^2 = 0$ Only $(0,0)$ satisfies

yz-plane $x=0$: $-2y^2 + 5z^2 = 0$, $z^2 = \frac{2}{5}y^2$, $z = \pm\sqrt{\frac{2}{5}}\,y$ two lines



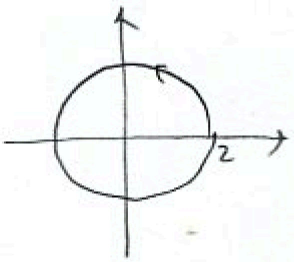


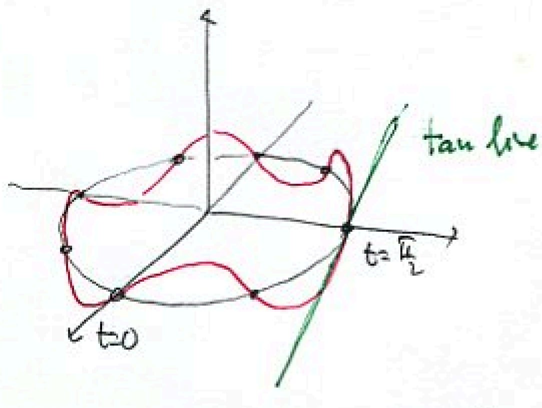


Cone centered on y-axis

6. (14pts) The curve $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \frac{1}{3}\sin(4t)\rangle$ is given, t any real number.
a) Sketch the curve in the coordinate system.
b) Find parametric equations of the tangent line to this curve when $t = \frac{\pi}{2}$ and sketch the tangent line.

a) $x = 2\cos t$
$y = 2\sin t$
is a circle
in xy-plane





$z = \frac{1}{3}\sin(4t)$ oscillates
4 full periods for one
revolution in xy-plane
$(0 \le t \le \pi)$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t, \frac{1}{3}\cos(4t)\cdot 4\rangle$

$\vec{r}(\frac{\pi}{2}) = \langle 0, 2, 0\rangle$

$\vec{r}'(\frac{\pi}{2}) = \langle -2, 0, \frac{4}{3}\rangle$

$$\boxed{\begin{aligned} x &= 0 - 2t \\ y &= 2 \\ z &= 0 + \tfrac{4}{3}t \end{aligned}}$$

eq. of tan line

7. (13pts) The points $A = (1, 3, -2)$ and $B = (4, -1, 3)$ are given.
a) Write parametric equations of the line segment AB.
b) Compute the length of the line segment using the parametrization and arc length formula.
c) Compare your answer in b) with the distance from A to B.

a) $\vec{AB} = \langle 3, -4, 5\rangle$

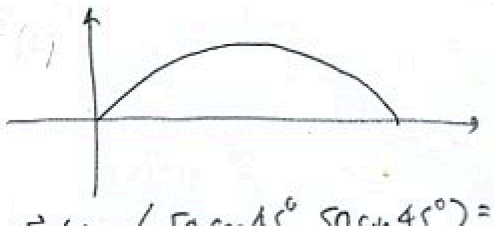
$\vec{r}(t) = \langle 1, 3, -2\rangle + t\langle 3, -4, 5\rangle$, $0 \le t \le 1$

b) $\ell = \int_0^1 |\vec{r}'(t)|\,dt = \int_0^1 |\langle 3, -4, 5\rangle|\,dt = \int_0^1 \sqrt{3^2 + (-4)^2 + 5^2}\,dt = \int_0^1 \sqrt{50}\,dt = 5\sqrt{2}$

c) dist. from A to B is $|\vec{AB}| = \sqrt{50} = 5\sqrt{2}$

8. (14pts) An arrow is launched from ground level at a 45° angle with initial speed 50 meters per second.
a) Assuming gravity acts in the usual negative y-direction (let $g = 10$), find the vector function $\mathbf{r}(t)$ representing the position of the arrow.
b) Find the range of the arrow.
c) Find the maximum height the arrow reaches.

a)



$\vec{v}(0) = \langle 50\cos 45°, 50\sin 45° \rangle = \langle 50\frac{\sqrt{2}}{2}, 50\frac{\sqrt{2}}{2} \rangle$

$= \langle 25\sqrt{2}, 25\sqrt{2} \rangle$

$\vec{r}(0) = \vec{0}$

$\vec{a}(t) = \langle 0, -10 \rangle$

$\vec{v}(t) = \langle 0, -10t \rangle + \vec{C}$

$\langle 25\sqrt{2}, 25\sqrt{2} \rangle = \vec{v}(0) = \vec{0} + \vec{C}$, so $\vec{C} = \langle 25\sqrt{2}, 25\sqrt{2} \rangle$

$\vec{v}(t) = \langle 25\sqrt{2}, 25\sqrt{2} - 10t \rangle$

$\vec{r}(t) = \langle 25\sqrt{2}t, 25\sqrt{2}t - 5t^2 \rangle + \vec{D}$

$\vec{0} = \vec{r}(0) = \vec{0} + \vec{D}$, so $\vec{D} = 0$

$\vec{r}(t) = \langle 25\sqrt{2}t, 25\sqrt{2}t - 5t^2 \rangle$

b) Hits ground when $y(t) = 0$

$25\sqrt{2}t - 5t^2 = 0$

$5t(5\sqrt{2} - t) = 0$

$t = 0, 5\sqrt{2}$

$x(5\sqrt{2}) = 25\sqrt{2} \cdot 5\sqrt{2} = 125 \cdot 2 = 250$ m

c) At highest point when $v_y(t) = 0$

$25\sqrt{2} - 10t = 0$

$t = \frac{25\sqrt{2}}{10} = \frac{5\sqrt{2}}{2}$

$y\left(\frac{5\sqrt{2}}{2}\right) = 25\sqrt{2} \cdot \frac{5\sqrt{2}}{2} - 5\left(\frac{5\sqrt{2}}{2}\right)^2$

$= 125 - 5 \cdot \frac{25}{2} = \frac{125}{2} = 62.5$ m

Bonus (10pts) Find the parametric equations of the line that is the intersection of the planes $x - y + 2z = 2$ and $x - y - 3z = 6$.

$x - y + 2z = 2$

$-(\,x - y - 3z = 6\,)$

$5z = -4$

$z = -\frac{4}{5}$

$y = -2 + x + 2z$

$= -2 + x - \frac{8}{5}$

$= x - \frac{24}{5}$

Use x as param.

$$\boxed{\begin{aligned} x &= t \\ y &= -\tfrac{24}{5} + t \\ z &= -\tfrac{4}{5} \end{aligned}}$$

Or: Find a pt. on line, set $x = 0$

$-y + 2z = 2$

$-(-y - 3z = 6)$

$5z = -4$

$z = -\frac{4}{5}$

$y = 2z - 2 = -\frac{8}{5} - 2$

$= -\frac{24}{5}$

Point: $(0, -\frac{24}{5}, -\frac{4}{5})$

$\vec{v} = \vec{n}_1 \times \vec{n}_2$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 1 & -1 & -3 \end{vmatrix}$

$= (3+2)\vec{i} - (-3-2)\vec{j} + (-1+1)\vec{k}$

$= 5\vec{i} + 5\vec{j}$, take $\vec{i} + \vec{j}$

Param. eq.: $x = s$, $y = -\frac{24}{5} + s$, $z = -\frac{4}{5}$