College Algebra — Lecture notes
MAT 140, Spring 2021 — D. Ivanšić

1.1 Introduction to Graphing

To describe points in the plane we use the rectangular coordinate system.

The coordinate axes divide the plane into four quadrants:

To graph an equation is to find all the points in the plane whose coordinates (x, y) satisfy the equation.

Example: Graph the equation $x^2 + 9y^2 = 25$ by finding pairs of numbers (x, y) that satisfy the equation.

Example: Graph the equation $2x - y + 1 = 0$. Use the fact that the equation can be easily solved for y.

The intersections of the graph with the x- and y-axes, are called the x- and y-intercepts. Find the x- and y-intercepts of the graph in previous example.

Example: Find the x- and y-intercepts of the graph of the equation $y = x^2 - 2x$ and use them and a few other points to sketch the graph.

Example: Sketch the graph of $y = x^3 - 3x^2 - 7x + 2$ using a graphing calculator.

Given points (x_1, y_1) and (x_2, y_2), we can find the distance between them.

Example. Draw a picture that helps you find the distance between points $(-2, 1)$ and $(3, 4)$.

The distance formula: The distance between points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by the formula

$$d = d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of two points.

On a number line:

In the coordinate plane:

The midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example. Find the midpoint M of the segment with endpoints $P_1 = (-2, 1)$ and $P_2 = (4, -3)$ and verify the distances from M to P_1 and P_2 are equal.

Definition. A *circle* is the set of all points in the plane that are a fixed distance r from a given point (h, k).

$r =$ radius of the circle, $(h, k) =$ center of the circle

From the definition and the distance formula, we can get the equation of a circle:

Equation of a circle with center (h, k) and radius r is (standard form):

$$(x - h)^2 + (y - k)^2 = r^2$$

Example. Write the equation of the circle with center $(2, 1)$ and radius 3.

Example. Graph the curve $(x + 5)^2 + (y - 3)^2 = 16$.

1.2 Functions and Graphs

Definition. A function is a correspondence (rule) between a set called a *domain* and a set called *range*, such that each element in the domain corresponds to (is sent to) exactly one element of the range.

If domain is X and range is Y, we write $f : X \to Y$ (f "sends" elements of X to elements of Y).

Example. Let f send a number to twice the number squared minus five times the number plus 3. We write:

$$f(x) =$$

$$x \mapsto$$

Find the following:

$f(1) =$

$f(-1) =$

$f(a) =$

$f(-x) =$

$f(x+1) =$

$f\left(\frac{x}{2}\right) =$

$x =$ input, independent variable $\qquad$ $f(x) =$ output, "value of f at x"

The graph of a function $f(x)$ is the graph of the equation $y = f(x)$, that is, the set of all points in the plane with coordinates $(x, f(x))$.

Note: Not every graph is the graph of a function.

$x^2 + y^2 = 16$

Vertical line test: If there is a vertical line that crosses the graph in more than one point, the graph is **not** the graph of a function. Otherwise, if every vertical line crosses the graph in at most one point, the graph is the graph of a function.

Example. Are these graphs of functions?

Example. A rectangle has width two inches less than length. Write its area as a function of length and state the domain of this function.

Example. Domain can be set:

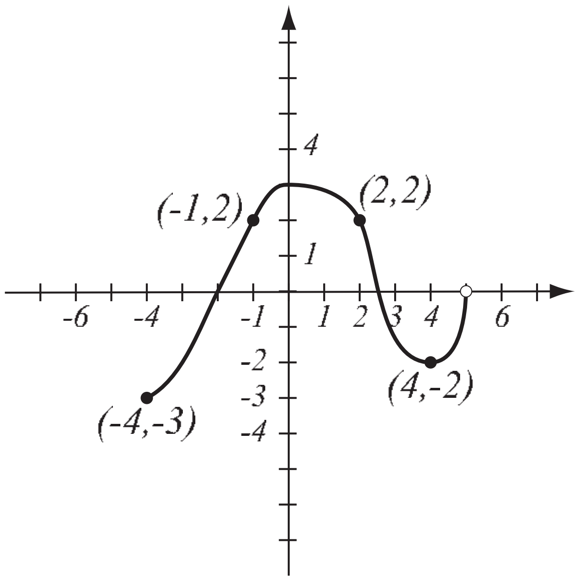
$f(x) = x^2 - x$, x in $[-5, 7] \cup [10, \infty)$

If domain is not set, it is the largest set of numbers x for which $f(x)$ is a real number.

Example. Find the domain of the function $g(x) = \dfrac{2x + 25}{x^2 + 4x - 5}$.

Example. Find the domain of the function $h(x) = \dfrac{\sqrt{x}}{x - 19}$.

Reading graphs



a) $f(-4) =$ $f(4) =$ $f(0) =$ $f(5) =$

b) Domain = set of all possible x-coordinates of points on graph =

c) Range = set of all possible y-coordinates of points on graph =

d) Find the x- and y-intercepts

e) Find all x such that a) $f(x) = 2$ b) $f(x) = -\frac{1}{2}$

1.3 Linear Functions

Definition. A function is *linear* if it can be written as

$$f(x) = mx + b, \text{where } m \text{ and } b \text{ are constant real numbers.}$$

When $m = 0$, $f(x) = b$ is a *constant function.*
When $m = 1$, $b = 0$, $f(x) = x$ is the *identity function.*

The graph of a linear function is a line.

Example. $f(x) = 2x - 1$ $\qquad$ $f(x) = x^2$

A linear function has the property: whenever x changes by the same amount, y changes by a constant amount. The ratio of these two is called *the slope of the line.*

$$\text{slope} = \frac{\text{rise}}{\text{run}} =$$

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example. Find the slope of the line passing through $(-1, 2)$ and $(2, -3)$. Draw several possibilities for rise and run.

Example. Slopes of lines are related to steepness of the line. Match the slope with the line.

$m = 0$ $\qquad m = \frac{1}{3}$ $\qquad m = 2$ $\qquad m = -\frac{1}{3}$ $\qquad m = -2$

Note: for vertical lines, slope is not defined.
The equation of a vertical line has form $x = a$.

Graphs of linear functions are graphs of the equation $y = mx + b$.

Set $x = 0$ to get $y = b$, hence b is the y-intercept of the line.
m is the slope of the line

Hence, the equation $y = mx + b$ is called the *slope-intercept* form of an equation of a line.

Example. Graph the line $y = \frac{1}{3}x - 1$. (Note: two points are enough to draw a line.)

Example. A road has a 4% grade. This means it rises 4ft for every 100ft of horizontal distance it covers.

Example. A cab company charges a \$1.95 start-up fee plus \$1.25 for every mile traveled.
a) Find the cost of a 15-mile ride.
b) Write the cost of a ride as a function of the number of miles traveled.

Example. In 2000, the population of Flint, MI was 124,943, and in 2010, it was 102,434.

The *average rate of change* of population over an interval of time is the slope of the line through the two data points at the endpoints of the interval. Thus,

$$\text{average rate of change of population} = \frac{\text{change in population}}{\text{change in time}} = \frac{y_2 - y_1}{x_2 - x_1}$$

which is the slope of the line through two points.

Similarly, we can define the average rate of change of any quantity y with respect to the quantity it depends on x as:

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

1.4 Equations of Lines

We find the equation of a line through point (x_1, y_1) with slope m.

$y - y_1 = m(x - x_1)$ *point-slope form of equation of a line* through point (x_1, y_1) with slope m

Example. Find the equation of the line through point $(3, 1)$ with slope 2. Draw the line.

Example. Find the equation of the line through points $(-1, -1)$ and $(1, 5)$. Draw the line.

The general equation of a line has form $Ax + By = C$.
This form includes vertical lines ($A = 1,\ B = 0,\ C = a$), which we cannot write in form $y = mx + b$.

Fact. Lines with slopes m_1 and m_2 are:
a) parallel if and only if $m_1 = m_2$
b) perpendicular if and only if $m_2 = -\frac{1}{m_1}$ (or: $m_1 \cdot m_2 = -1$)

Example. Find the equation of the line through $(-2, 3)$ that is
a) parallel to line $2x - 4y = 7$
b) perpendicular to line $2x - 4y = 7$.
Draw all the lines.

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1.4 Modeling: Line of best fit

Example. From past experience a marketing company has assembled the following table that relates advertising expenses A and sales S (amounts in thousands).
a) Draw the scatterplot of the data. Does the relationship look linear?
b) Use two points in the scatterplot to get an equation of a line that models the relationship between A and S.
c) Use the "line of best fit" method to find a line whose combined squares of errors is smallest. This line is considered to best model a linear relationship arising from a scatterplot.
d) Find the coefficient of correlation r. How strong is the linear relationship between A and S?
e) What sales should we expect with advertising expenses A=26?

A	S
20	335
22	339
22.5	338
24	343
24	341
27	350
28.3	351

1.5 Linear equation applications

Example. The length of a rectangle is 4cm more than the width. If the width is increased by 1cm and the length is increased by 5cm, the new rectangle has perimeter 32cm. What are the dimensions of the original rectangle?

Example. Judy and Tom agree to share the cost of an \$18 pizza based on how much each ate. If Tom ate 2/3 of the amount that Judy ate, how much should each pay?

Example (interest). Wendy, a loan officer at a bank, has \$1,000,000 to lend and is required to obtain an average annual return of 18%. If she can lend at the rate of 19% or at the rate 16%, how much can she lend at the 16% rate and still meet her requirement?
Recall formula: $I = Prt$ *(interest = principal · rate · time).*

Example (uniform motion). In the morning, Morgan drove to a business appointment at 50mph. Due to traffic, his average speed on the return was 40mph, so the return took a quarter of an hour longer. How far did Morgan travel for his appointment?
Recall formula: $d = rt$ *(distance = rate · time).*

Example (mixture). A painter needs a 10% solution of blue paint for a job. How much of a 5% solution of blue should be mixed with 60 liters of a 20% solution of blue to get a 10% solution?
Solution principle: consider how much pure blue there is before and after mixing.

Methods of attack of these problems:
1) Familiarize yourself with the problem: read it carefully (more than once, if needed) What is given, and what are you asked to find: assign a variable (x, t, u, etc.) to represent what you are asked to find. Write down what it represents. If applicable, draw the @!♣#◇$ picture! Large, clear, with indicated quantities.
2) Write an equation that represents the facts of the problem, either from the picture or from your notes.
3) Solve the equation.
4) Reality check: does your answer make sense? Check with facts of problem.

1.6 Solving Linear Inequalities

Example. Solve the inequalities.

$3x + 3 < 5 - x$

$4 - 3x \leq 7$

Example. Solve the double inequality $7 < 3x - 5 \leq 9$.

A double inequality is two inequalities that must both be satisfied, that is

$7 < 3x - 5$ and $3x - 5 \leq 9$

If x appears only in the middle, we can solve both inequalities at once:

$7 < 3x - 5 \leq 9$

Example. Inequalities may also be joined by the conjunction "or".

$2x - 1 \leq -1 \qquad \text{or} \qquad 2x - 1 > 3$

Example. Find the domain of $f(x) = \sqrt{3x - 2}$.

Example. Henry is comparing rental car offers from two companies:
Company A charges \$20 per day plus \$0.21 per mile.
Company B charges \$30 per day plus \$0.13 per mile.
For which mileage traveled is a one-day rental with company A better?