Calculus 2 — Exam 0
MAT 308, Fall 2021 — D. Ivanšić

Name: ______________________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dx}\left(3x^4 - e^6 + \sqrt[3]{x^5} + \dfrac{4}{x^7}\right) =$

2. (6pts) $\dfrac{d}{dx}\left(e^{2x}\cos^3 x\right) =$

3. (8pts) $\dfrac{d}{dt}\dfrac{\left(t^2-1\right)^3}{\left(t^2+1\right)^2} =$

4. (4pts) $\dfrac{d}{dx}\dfrac{1}{x\ln x} =$

5. (6pts) $\dfrac{d}{d\theta}\arcsin^2(\tan\theta) =$

6. (7pts) Find the first and second derivatives of $f(x) = \cos\left(x^2\right)$.

7. (5pts) Let $f(x) = \dfrac{1}{x^3}$. Take the first four derivatives of f, and try to spot the pattern. What is $f^{(20)}(x)$, the 20th derivative of f? How about $f^{(n)}(x)$?

Find the following limits. Use L'Hospital's rule if needed.

8. (2pts) $\lim\limits_{x \to 0-} \dfrac{1}{x^5} =$

9. (6pts) $\lim\limits_{x \to \infty} \dfrac{x^3 - 5x + 4}{-3x^2 + x + 2} =$

10. (8pts) $\lim\limits_{x \to \infty} (1 + \ln x)^{\frac{1}{x}} =$

Find the following antiderivatives.

11. (7pts) $\displaystyle\int 2x^4 - \frac{2}{1+x^2} + \sqrt[5]{x^6} + c^2\,dx =$

12. (3pts) $\displaystyle\int \sin\left(4x - \frac{\pi}{4}\right)\,dx =$

13. (7pts) $\displaystyle\int \frac{\sqrt{x} + \sqrt[3]{x}}{\sqrt[6]{x}}\,dx =$

Use the substitution rule in the following integrals:

14. (7pts) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx =$

15. (10pts) $\displaystyle\int_{e^8}^{e^{64}} \frac{dx}{x\sqrt[3]{\ln x}} =$

16. (8pts) Consider the integral $\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos x \, dx$.

a) Draw a picture to explain the meaning of the integral.
b) Use the picture to estimate whether the integral is positive or negative.
c) Evaluate the integral to verify your finding in b).

Bonus. (10pts) The rear inside cover of our book claims that

$$\int \frac{dx}{x^2 - a^2}\, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Verify this formula by differentiating.

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Show all your work!

Find the following integrals:

1. (12pts) $\displaystyle\int e^x \cos x \, dx =$

2. (8pts) $\displaystyle\int \sin^2 x \, \cos^2 x \, dx =$

Determine whether the following improper integral converges by calculating it directly.

3. (10pts) $\displaystyle\int_0^\infty \frac{\arctan x}{1+x^2} \, dx =$

Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (14pts) $\int x^3 \sqrt{1-x^2}\, dx =$

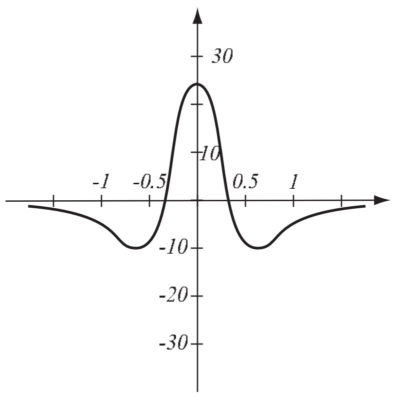
5. (14pts) $\int_0^3 \frac{1}{(9+x^2)^{\frac{3}{2}}}\, dx =$

Use the method of partial fractions to find the integral.

6. (14pts) $\displaystyle\int \frac{-x^2+2x+3}{(x-1)^3}\,dx =$

7. (10pts) Use comparison to determine whether the improper integral $\displaystyle\int_2^{\infty} \frac{x^5}{x^6-4}\,dx$ converges.

8. (18pts) Suppose we wanted to approximate the number $\frac{\pi}{4} = \arctan 1$. We could do it by approximating the integral $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$, which uses only the four algebraic operations.
a) Write the expression you would use to calculate S_6, the Simpson rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) Find the error estimate for S_n in general. The graph of the fourth derivative of $\frac{1}{1+x^2}$, which is $\frac{24(5x^4-10x^2+1)}{(1+x^2)^5}$, is shown in the picture.
c) Estimate the error for S_6.
d) What should n be in order for S_n to give you an error less than 10^{-8}?



Bonus (10pts) Determine for which $p > 0$ the integral below converges. (Note this is not the standard knowledge one because the interval is different.)

$$\int_0^1 \frac{1}{x^p}\,dx$$

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Show all your work!

1. (24pts) The region bounded by the curves $y = 5 - x^2$ and $y = x + 3$ is rotated around the x-axis.
a) Sketch the solid and a typical cross-sectional washer.
b) Set up the integral for the volume of the solid.
c) Evaluate the integral.

2. (14pts) Consider the triangle bounded by the x-axis and lines $y = \dfrac{1}{2}x + 2$ and $y = x - 2$.
a) Sketch the triangle.
b) Set up the integral that computes its area. Simplify, but do not evaluate the integral.

3. (16pts) Rotate the region bounded by the curve $y = \sin x$ for $0 \leq x \leq \pi$ and the x-axis about the line $x = -1$ to get a solid.

a) Sketch the solid and a typical cylindrical shell.

b) Set up the integral for the volume of the solid using the shell method. Simplify, but do not evaluate the integral.

4. (18pts) The base of a solid is the region in the xy-plane bounded by curves $y = \sqrt{x}$, $x = 4$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are right triangles, where one side lies on the base, the other is perpendicular to the base, and the hypothenuse makes angle $\frac{\pi}{6}$ with the side lying in the base.

a) Sketch the solid and a typical cross-section.

b) Set up the integral for the volume of the solid.

c) Evaluate the integral.

5. (12pts) Set up the integral for the length of the curve $y = e^x - x$ from $x = 0$ to $x = 3$. Simplify, but do not evaluate the integral.

6. (16pts) On a building site in heavy rain, a bucket weighing 50kg is lifted 60 meters. The amount of water in the bucket is 100 liters at bottom and 120 liters by the time it reaches the top. Set up the integral for the work needed to lift the bucket. (Ignore cable weight.) Assume $g = 10$ and water density = 1kg/liter. Simplify, but do not evaluate the integral.

Bonus (10pts) Let $0 < a < b$. By rotating the circle of radius a centered at the origin around the line $y = b$, you will get a torus (surface of a doughnut). Set up the integral for the surface area of the torus. (Tip: exploit symmetry about the y-axis for a simpler solution.) Do not evaluate the integral.

Calculus 2 — Exam 3
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Show all your work!

Find the limits, if they exist.

1. (6pts) $\lim\limits_{n\to\infty} (-1)^n \dfrac{2^{2n+3}}{5^n} =$

2. (6pts) $\lim\limits_{n\to\infty} \tan \dfrac{(4n+1)\pi}{4} =$

3. (10pts) Find the limit. Use the theorem that rhymes with the blank from "______ Navidad."

$$\lim_{n\to\infty} \frac{5+(-1)^n}{n^4+3n^2}$$

4. (6pts) Write the series using sigma notation:

$$\frac{7}{5} - \frac{9}{25} + \frac{11}{125} - \frac{13}{625} + \cdots =$$

5. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{5 \cdot 3^{2n+3}} =$$

Determine whether the following series converge and justify your answer.

6. (12pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+5n-3}$

7. (6pts) $\displaystyle\sum_{n=1}^{\infty} \left(\sin\frac{1}{n} - \sqrt[n]{2} \right)$

8. (8pts) Consider the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$.

a) Is the series convergent? Justify.
b) Is the series absolutely convergent? Justify.

9. (12pts) For the sequence $\left\{\frac{n}{(n+5)^2}\right\}_{n=1}^{\infty}$, determine:
a) for which n the sequence is decreasing.
b) its upper and lower bounds.

Determine whether the following series converge using the root or ratio test.

10. (11pts) $\sum_{n=2}^{\infty}\frac{2^{2n}(n^2+n+7)}{3^{n+1}(n^3-n^2)}$

11. (11pts) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^{11}}{4^n \cdot n!}$

Bonus. (10pts) Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$below.

a) Show that $\lim_{n\to\infty} b_n = 0$ by considering the odd and even terms separately.

b) Show that the sequence b_n is not decreasing.

c) Show that the partial sum s_{2n} of the series below satisfies $s_{2n} = u_n - v_n$, where u_n, v_n are partial sums of familiar series.

d) Use c) to help you answer: does the series below converge?

$$\frac{1}{1} - \frac{1}{2^1} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{2^3} + \frac{1}{4} - \frac{1}{2^4} + \cdots =$$

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Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (16pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n^2} \cdot (x-1)^n$

2. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}\, x^n$

3. (6pts) Use a known power series to find the sum. It's not a typo — it really is $(2n)!$ in the denominator, not $(2n+1)!$ Think.

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{3^{2n+1}(2n)!} =$$

4. (8pts) Use a known power series to find the limit.

$$\lim_{x\to 0} \frac{\ln(1+3x^3) - 3x^3}{x^6} =$$

5. (14pts) Use geometric series to get a power series for $\dfrac{2x-8}{x^2-8x+15}$. The partial fraction decomposition has been written for you. Your answer needs to be a single sum of type $\sum c_n x^n$. State the interval of convergence (no need to check the endpoints).

$$\frac{2x-8}{x^2-8x+15} = \frac{1}{x-5} + \frac{1}{x-3} =$$

6. (12pts) Use a geometric series and antidifferentiation to find the McLaurin series for $\arctan x$.

7. (18pts) Let $f(x) = \ln x$.
a) Find the 3nd Taylor polynomial for f centered at $a = 4$.
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $[3, 5]$. Leave your answer as a fraction.

8. (16pts) Use the known power series for $\sin x$ to find the series representing $\int_0^1 \sin(x^2)\,dx$. (Note that $\sin(x^2)$ does not have an antiderivative that is an elementary function.) Give an approximation of the definite integral with accuracy 10^{-4}. Write the approximation as a sum (you do not have to simplify it).

Bonus (10pts) Find a fraction that is the approximation of e with accuracy 10^{-3}. Use the series for e^x and Taylor's formula, and assume you know $e < 3$. Write the approximation as a sum (you do not have to simplify it).

Calculus 2 — Exam 5
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Name:____________________________

Show all your work!

1. (12pts) Polar coordinates of two points are given.
a) Sketch the points in the plane.
b) For each point, give two additional polar coordinates, one with a negative r, one with a negative θ.

$\left(-1, \dfrac{3\pi}{4}\right)$ $\qquad\qquad$ $\left(3, \dfrac{7\pi}{5}\right)$

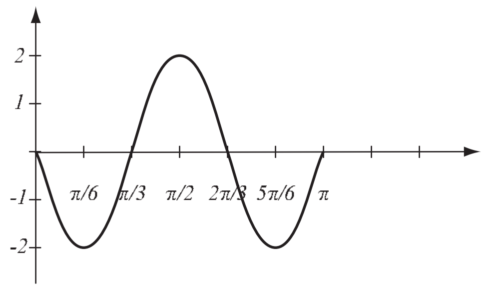
2. (10pts) Convert (a picture may help):
a) $\left(8, \dfrac{2\pi}{3}\right)$ from polar to rectangular coordinates
b) $(-2\sqrt{3}, 2)$ from rectangular to polar coordinates

3. (14pts) Find the equation of the tangent line to the parametric curve $x = \dfrac{\cos t}{t}$, $y = \dfrac{\sin t}{t}$ at the point where $t = \pi$.

4. (12pts) A particle moves along the path with parametric equations $x(t) = \sin t$, $y(t) = 3 + \cos^2 t$ for $0 \le t \le 2\pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

5. (6pts) Identify the curve given in polar coordinates by $r = 5 \sec\theta$ by converting the equation to cartesian coordinates.

6. (12pts) The graph of $r = f(\theta)$ is given in cartesian coordinates. Use its intervals of increase and decrease to help you sketch the polar curve $r = f(\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.



7. (16pts) The parametric curve $x = 3 + t$, $y = 1 + 2t$, $-1 \leq t \leq 2$ is given.
a) Use an integral to find the length of the curve.
b) Eliminate the parameter in order to find out what the curve is.
c) Use a college algebra method to find its length and compare your answer to a).

8. (18pts) A parametric curve is given by $x(t) = 2t^3 - 3t^2$, $y(t) = t^2 - 6t + 2$.
a) Find the points on the curve where the tangent line is horizontal or vertical.
b) Where does the curve go as $t \to \infty$ and $t \to -\infty$? (That is, find $\lim_{t \to \pm\infty} x(t)$, $\lim_{t \to \pm\infty} y(t)$.)
c) Plot the points from a) on a coordinate system and use them, along with information from b), or from plotting additional points, to get a graph of the curve. Recall that the curve moves in only one of general directions $\nearrow \nwarrow \swarrow \searrow$ between points from a).

Bonus. (10pts) Show that the length of a polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is given by the formula below. *Hint: to start, get a parametrization* $(x(\theta), y(\theta))$ *of the polar curve by using formulas for the polar* $\to$ *cartesian conversion.*

$$l = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

Calculus 2 — Final Exam
MAT 308, Fall 2021 — D. Ivanšić

Name: ______________________________

Show all your work!

Find the following integrals:

1. (6pts) $\displaystyle\int xe^{2x}\,dx =$

2. (10pts) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \sin^3 x\,dx =$

3. (12pts) Determine whether the following improper integral converges by calculating it directly.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx =$$

4. (10pts) Convert (a picture may help):
a) $\left(4, \dfrac{5\pi}{4}\right)$ from polar to rectangular coordinates
b) $(3, -3\sqrt{3})$ from rectangular to polar coordinates

5. (24pts) The region bounded by the curves $y = x^2 + 1$ and $y = 5$ is rotated around the x-axis.
a) Sketch the solid and a typical cross-sectional washer.
b) Set up the integral for the volume of the solid.
c) On another picture, sketch the solid and a typical cylindrical shell.
d) Set up the integral for the volume of the solid using the shell method.
Simplify, but do not evaluate the integrals.

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5\cdot 3^{2n+1}}{16^n} =$$

7. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{4^{n+1}\cdot(n+3)}$$

8. (18pts) Let $f(x) = \sqrt{x}$.
a) Find the 3rd Taylor polynomial for f centered at $a = 9$.
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $(6, 12)$.

9. (10pts) A particle moves along the path with parametric equations $x(t) = 3 + \sin^2 t$, $y(t) = 1 - \cos^2 t$, $0 \le t \le \pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

10. (24pts) The integral $\int_0^1 \sin(x^2)\, dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $f(x) = \sin(x^2)$ is not expressible using elementary functions.
a) Write the expression you would use to calculate S_6, the Simpson rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) It is known that $-29 < f^{(4)}(x) < 0$ on $[0, 1]$: use it to find the error estimate for S_n in general.
c) What should n be in order for S_n to give you an error less than 10^{-4}?
d) Use the known power series for $\sin x$ to find a power series for the above integral.
e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).
f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, Simpson rule or series?

11. (12pts) First draw the graph of $r = \cos\theta - \frac{1}{2}$ in a cartesian θ-r coordinates. Use this graph to draw the polar curve with the same equation.

Bonus (15pts) Find a fraction that is the approximation of e with accuracy 10^{-4}. Use the series for e^x and Taylor's formula, and assume you know $e < 3$. Write the approximation as a sum (you do not have to simplify it).