Calculus 2 — Final Exam MAT 308, Fall 2021 — D. Ivanšić

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Find the following integrals:

1. (6pts) $\int xe^{2x}\,dx = \begin{bmatrix} u=x & dv=e^{2x}dx \\ du=dx & v=\frac{e^{2x}}{2} \end{bmatrix} = x\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

2. (10pts) $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^3 x\,dx = \begin{bmatrix} u=\cos x\,dx & x=\frac{\pi}{2},\ u=0 \\ du=-\sin x\,dx & x=0,\ u=1 \end{bmatrix} = \int_1^0 u^3(1-u^2)(-du)$

$\cos^3 x \underbrace{\sin^2 x}_{1-\cos^2 x} \sin x\,dx$ $\quad -du = \sin x\,dx$

$$= \int_0^1 u^3 - u^5\,du = \left(\frac{u^4}{4} - \frac{u^6}{6}\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

3. (12pts) Determine whether the following improper integral converges by calculating it directly.

$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx = \begin{bmatrix} u=\ln x & dv=\frac{1}{x^2}=x^{-2} \\ du=\frac{1}{x}dx & v=\frac{x^{-1}}{-1} \end{bmatrix}$

$\int_1^t$

$$= \lim_{t\to\infty}\left(-x^{-1}\ln x\Big|_1^t + \int_1^t \underbrace{\frac{1}{x}\cdot x^{-1}}_{x^{-2}}\,dx\right) = \lim_{t\to\infty}\left(-\frac{\ln x}{x}\Big|_1^t + (-x^{-1})\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(-\left(\frac{\ln t}{t} - 0\right) - \left(\frac{1}{t} - 1\right)\right) = \lim_{t\to\infty}\left(1 - \underbrace{\frac{1}{t}}_{\to 0} - \frac{\overset{\to\infty}{\ln t}}{\underset{\to\infty}{t}}\right) \overset{LH}{=}$$

$$1 - \lim_{t\to\infty}\underbrace{\left(\frac{\frac{1}{t}}{1}\right)}_{\to 0} = 1 - 0 = 1$$

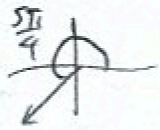
4. (10pts) Convert (a picture may help):

a) $\left(4, \dfrac{5\pi}{4}\right)$ from polar to rectangular coordinates

b) $(3, -3\sqrt{3})$ from rectangular to polar coordinates

a) $x = r\cos\theta = 4 \cdot \cos\frac{5\pi}{4} = 4\cdot\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y = r\sin\theta = 4\sin\frac{5\pi}{4} = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$



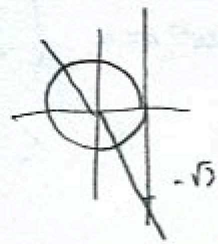
Cartesian coordinates:

$(-2\sqrt{2}, -2\sqrt{2})$

b) $\tan\theta = \dfrac{-3\sqrt{3}}{3} = -\sqrt{3}$

$\theta = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$

correct quadrant



$r = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{9 + 9\cdot 3}$

$= \sqrt{36} = 6$

Polar coord: $\left(6, -\frac{\pi}{3}\right)$

5. (24pts) The region bounded by the curves $y = x^2 + 1$ and $y = 5$ is rotated around the x-axis.

a) Sketch the solid and a typical cross-sectional washer.

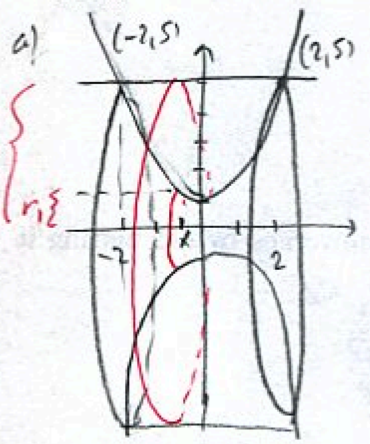
b) Set up the integral for the volume of the solid.

c) On another picture, sketch the solid and a typical cylindrical shell.

d) Set up the integral for the volume of the solid using the shell method.

Simplify, but do not evaluate the integrals.

a)

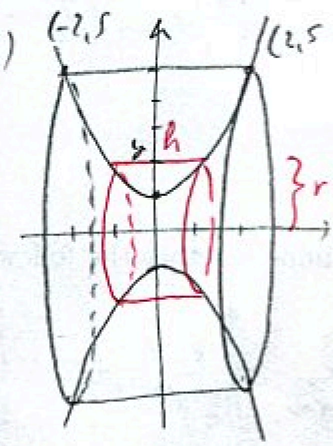


$x^2 + 1 = 5$

$x^2 = 4$

$x = \pm 2$

c)



$S(y) = 2\pi r h$

$= 2\pi y\left(\sqrt{y-1} - (-\sqrt{y-1})\right)$

$= 2\pi y \cdot 2\sqrt{y-1}$

$y = x^2 + 1$

$y - 1 = x^2$

$x = \pm\sqrt{y-1}$

x

b) $A(x) = \pi(r_2^2 - r_1^2)$

$= \pi\left(5^2 - (x^2+1)^2\right)$

$V = \int_{-2}^{2} \pi\left(5^2 - (x^2+1)^2\right) dx$

$= \pi \int_{-2}^{2} 25 - (x^4 + 2x^2 + 1)\, dx$

$= \pi \int_{-2}^{2} 24 - x^4 - 2x^2\, dx$

d) $V = \int_{1}^{5} 2\pi y \cdot 2\sqrt{y-1}\, dy$

$= 4\pi \int_{1}^{5} y\sqrt{y-1}\, dy$

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5\cdot 3^{2n+1}}{16^n} = \sum_{n=1}^{\infty}(-1)\cdot 5\cdot 3\cdot \frac{(-1)^n 3^{2n}}{16^n} = \sum_{n=1}^{\infty}(-15)\left(-\frac{9}{16}\right)^n$$

$$= \left[\begin{array}{l}\text{geometric series}\\ \left|-\frac{9}{16}\right|<1, \text{ so converges}\end{array}\right] = \frac{\text{first term}}{1-r} = \frac{(-1)^2\cdot\frac{5\cdot 3^3}{16}}{1-\left(-\frac{9}{16}\right)}$$

$$= \frac{\cancel{5}\cdot 27}{\cancel{16}}\cdot\frac{\cancel{16}}{\underset{5}{\cancel{25}}} = \frac{27}{5}$$

7. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{4^{n+1}\cdot(n+3)}$$

Root test: $\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(x-2)^n}{4^{n+1}(n+3)}\right|} = \frac{\sqrt[n]{|x-2|^n}}{\sqrt[n]{4^n}\,\sqrt[n]{4(n+3)}} = \frac{|x-2|}{4\sqrt[n]{4(n+3)}} \to \frac{|x-2|}{4}$

$\sqrt[n]{\text{polynomial}(n)} \to 1$

Must have $\frac{|x-2|}{4}<1$

$|x-2|<4$

$-2 < x < 6$

$x=6$ gives $\sum_{n=1}^{\infty}\frac{4^n}{4^{n+1}(n+3)} = \sum_{n=1}^{\infty}\frac{1}{4(n+3)}$ diverges, like $\sum\frac{1}{n}$

$x=-2$ gives $\sum_{n=1}^{\infty}\frac{(-4)^n}{4^{n+1}(n+3)} = \sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{4^{n+1}(n+3)} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{4(n+3)}$

$\frac{1}{4(n+3)}\to 0$ and sequence is decreasing, so series converges by alternating series test.

Interval of convergence:

$[-2,6)$

$$\frac{24\cdot 27}{\begin{array}{r}48\\168\\648\cdot 6\end{array}} \quad 3888$$

8. (18pts) Let $f(x) = \sqrt{x}$.
a) Find the 3rd Taylor polynomial for f centered at $a = 9$.
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $(6, 12)$.

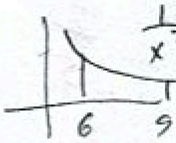
a)

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$1x^{\frac{1}{2}}$	3
1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{2}9^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{4}9^{-\frac{3}{2}} = -\frac{1}{4}\cdot\frac{1}{27} = -\frac{1}{108}$
3	$\frac{3}{8}x^{-\frac{5}{2}}$	$\frac{3}{8}9^{-\frac{5}{2}} = \frac{\cancel{3}}{8}\cdot\frac{1}{27\cdot\cancel{9}_{3}} = \frac{1}{648}$
4	$-\frac{15}{16}x^{-\frac{7}{2}}$	

$$T_3(x) = 3 + \frac{\frac{1}{6}}{1!}(x-9) - \frac{\frac{1}{108}}{2!}(x-9)^2 + \frac{\frac{1}{648}}{3!}(x-9)^3$$

$$= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3$$

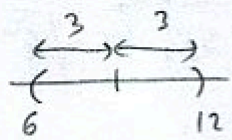
$\frac{1}{x^{7/2}}$ has max at $x=6$



b)
$$|R_3(x)| = \left|\frac{f^{(4)}(z)}{4!}(x-9)^4\right| = \frac{\left|-\frac{15}{16}\frac{1}{z^{7/2}}\right|}{24}\underbrace{|x-9|}_{\le 3 \text{ on}}{}^4 \le \frac{\overset{5}{\cancel{15}}}{16\cdot\cancel{24}_8}\cdot\frac{1}{6^{7/2}}\cdot 3^4$$

$$\le \frac{3^4}{128\cdot 6^{7/2}} \le \frac{\cancel{3^4}^{\,3}}{128\cdot 6^3} = \frac{3}{1024}$$

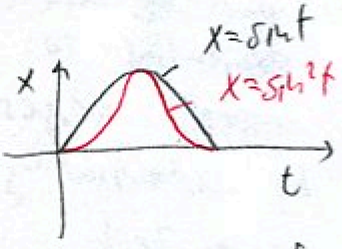
$6^3 \le 6^{\frac{7}{2}}$ $\quad 2^3 3^3$

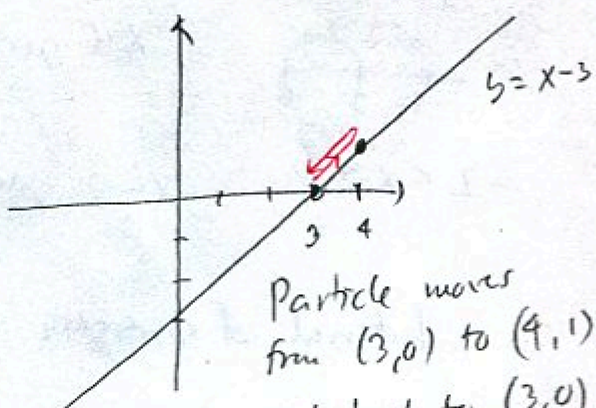


9. (10pts) A particle moves along the path with parametric equations $x(t) = 3 + \sin^2 t$, $y(t) = 1 - \cos^2 t$, $0 \le t \le \pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = 3 + \sin^2 t \Rightarrow \sin^2 t = x - 3$
$y = 1 - \cos^2 t = \sin^2 t = x - 3$

Path is some part of $y = x - 3$





Particle moves from $(3,0)$ to $(4,1)$ and back to $(3,0)$ along line $y = x-3$

When t goes from 0 to π
$\sin^2 t$ goes from 0 to 1 and back to 0
So $x = 3 + \sin^2 t$ goes from 3 to 4 and back to 3

10. (24pts) The integral $\int_0^1 \sin(x^2)\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $f(x) = \sin(x^2)$ is not expressible using elementary functions.
a) Write the expression you would use to calculate S_6, the ~~midpoint~~ Simpson rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) It is known that $-29 < f^{(4)}(x) < 0$ on $[0,1]$: use it to find the error estimate for S_n in general.
c) What should n be in order for S_n to give you an error less than 10^{-4}?
d) Use the known power series for $\sin x$ to find a power series for the above integral.
e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).
f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, Simpson rule or series?

a)

$$S_6 = \frac{\frac{1}{6}}{3}\left(f(0)+4f(\tfrac{1}{6})+2f(\tfrac{1}{3})+\dots+f(1)\right)$$

$0 \quad \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1$

$$= \frac{1}{18}\left(\sin 0^2 + 4\sin\frac{1}{36} + 2\sin\frac{1}{9} + 4\sin\frac{1}{4} + 2\sin\frac{4}{9} + 4\sin\frac{25}{36} + \sin 1\right)$$

b) $-29 < f^{(4)}(x) \le 0$

so $|f^{(4)}| \le 29$

$$|E_S| \le \frac{K(1-0)^5}{180\cdot n^4} = \frac{29}{180n^4} \le \frac{1}{6n^4}$$

c) $\frac{29}{180n^4} \le 10^{-4}$ $\quad \frac{10^4}{6} \le n^4$ $\quad n \ge \sqrt[4]{\frac{10^4}{6}} = \frac{10}{\sqrt[4]{6}} \ge 6.38$ $\quad n \ge 8$

d)

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^{4n+2}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$$

alternating series an decreasing may use alt. series estimate.

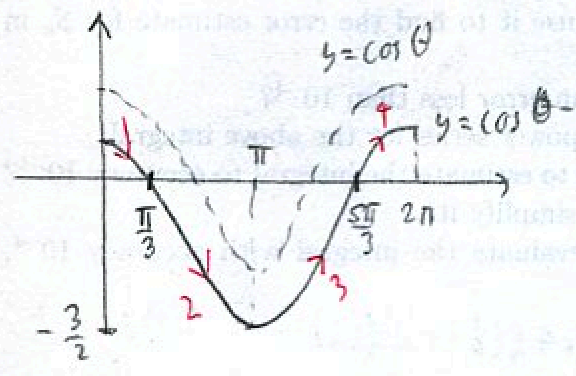
e) need $(4n+3)(2n+1)! \ge 10^4$

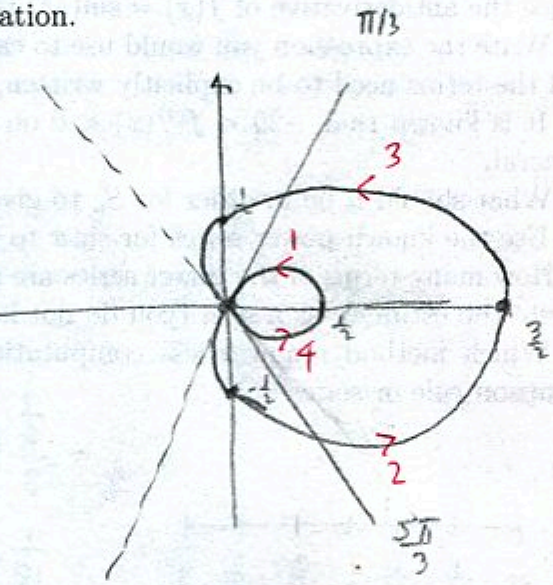
n	$(4n+3)(2n+1)!$
2	$11\cdot 5! = 11\cdot 120 < 10^4$
3	$15\cdot 7! = 15\cdot 120\cdot 42 = 630\cdot 120 > 10^4$

$$S_2 = \frac{1}{3\cdot 1!} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320}$$

f) Series needs three divisions and two additions, while Simpson rule would require at least 8 evaluations of $\sin x$, each with multiple multiplications and additions. Series is better.

11. (12pts) First draw the graph of $r = \cos\theta - \frac{1}{2}$ in a cartesian θ-r coordinates. Use this graph to draw the polar curve with the same equation.





Bonus (15pts) Find a fraction that is the approximation of e with accuracy 10^{-4}. Use the series for e^x and Taylor's formula, and assume you know $e < 3$. Write the approximation as a sum (you do not have to simplify it).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

On interval $[0,1]$, $0 \le x \le 1$

$0 \le z \le x \le 1$

$e^z \le e^1 < 3$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \cdot x^{n+1} \right| = \frac{e^z}{(n+1)!} |x|^{n+1} \le \frac{3}{(n+1)!} 1^{n+1} = \frac{3}{(n+1)!}$$

Need $\frac{3}{(n+1)!} \le 10^{-4}$ $\quad (n+1)! \ge 3 \cdot 10^4$

n	n!
5	120
6	720
7	5040
8	$40,160 \ge 3 \cdot 10^4$

$$S_7 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = \frac{13700}{5040}$$

$$\simeq 2.718254$$

approximates e with accuracy 10^{-4}

(calculator $e = 2.718282$, error about 3×10^{-5}