Calculus 2 — Exam 5
MAT 308, Fall 2021 — D. Ivanšić

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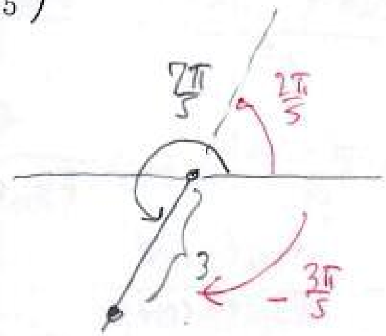
Show all your work!

1. (12pts) Polar coordinates of two points are given.
a) Sketch the points in the plane.
b) For each point, give two additional polar coordinates, one with a negative r, one with a negative θ.

$\left(-1, \frac{3\pi}{4}\right)$ $\left(3, \frac{7\pi}{5}\right)$



$\left(-1, \frac{3\pi}{4}+2\pi\right) = \left(-1, \frac{11\pi}{4}\right)$

$\left(1, -\frac{\pi}{4}\right)$

$\left(-3, \frac{2\pi}{5}\right)$

$\left(3, -\frac{3\pi}{5}\right)$

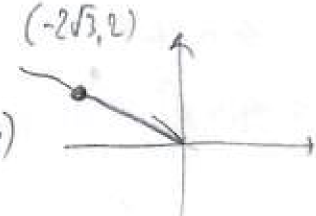
2. (10pts) Convert (a picture may help):
a) $\left(8, \frac{2\pi}{3}\right)$ from polar to rectangular coordinates
b) $(-2\sqrt{3}, 2)$ from rectangular to polar coordinates

a) $x = r\cos\theta = 8\cos\frac{2\pi}{3} = 8\cdot\left(-\frac{1}{2}\right) = -4$

$y = r\sin\theta = 8\sin\frac{2\pi}{3} = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$

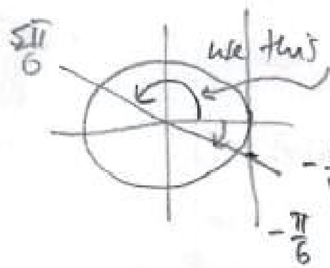
$(-4, 4\sqrt{3})$

b)



$r = \sqrt{(-2\sqrt{3})^2 + 2^2}$
$= \sqrt{4\cdot 3 + 4} = \sqrt{16} = 4$

$\tan\theta = \frac{2}{-2\sqrt{3}} = -\frac{1}{\sqrt{3}}, \ \theta = \frac{5\pi}{6}$



use this angle $\left(4, \frac{5\pi}{6}\right)$

3. (14pts) Find the equation of the tangent line to the parametric curve $x = \dfrac{\cos t}{t}$, $y = \dfrac{\sin t}{t}$ at the point where $t = \pi$.

$$x' = \frac{-\sin t \cdot t - \cos t \cdot 1}{t^2} = -\frac{t\sin t + \cos t}{t^2}$$

$$y' = \frac{\cos t \cdot t - \sin t \cdot 1}{t^2} = \frac{t\cos t - \sin t}{t^2}$$

$$\frac{y'}{x'} = \frac{t\cos t - \sin t}{t^2} \cdot (-1)\frac{t^2}{t\sin t + \cos t}$$

$$\frac{y'}{x'} = -\frac{t\cos t - \sin t}{t\sin t + \cos t}$$

At $t=\pi$, $\dfrac{y'}{x'} = -\dfrac{\pi(-1) - 0}{\pi \cdot 0 - 1} = -\pi$

$x(\pi) = \dfrac{-1}{\pi}$ $\quad y(\pi) = 0$

Eq. of tan. line:

$$y - 0 = -\pi\left(x - \left(-\tfrac{1}{\pi}\right)\right)$$

$$y = -\pi x - 1$$

4. (12pts) A particle moves along the path with parametric equations $x(t) = \sin t$, $y(t) = 3 + \cos^2 t$ for $0 \le t \le 2\pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

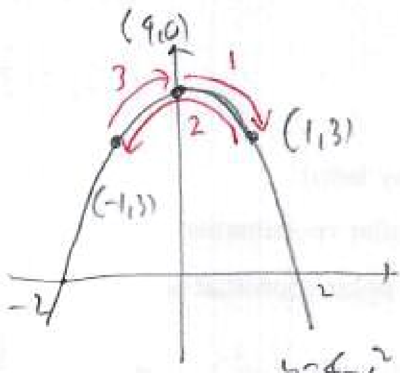
$x = \sin t$

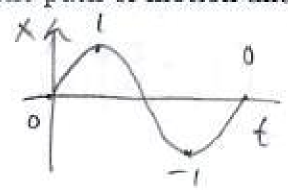
$y = 3 + \cos^2 t$

$= 3 + 1 - \sin^2 t$

$= 4 - \sin^2 t$

$y = 4 - x^2$





For $0 \le t \le 2\pi$,

x goes from 0 to 1

then 1 to -1

then -1 to 0

Particle moves from (4,0) to (1,3) to (-1,3) to (4,0) along $y = 4 - x^2$

5. (6pts) Identify the curve given in polar coordinates by $r = 5\sec\theta$ by converting the equation to cartesian coordinates.

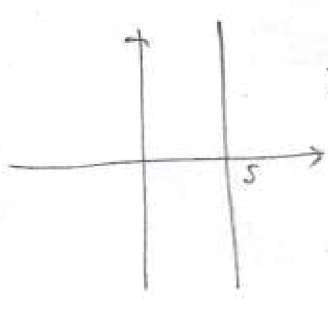
$r = 5\sec\theta$

$r = \dfrac{5}{\cos\theta}$

$r\cos\theta = 5$

$x = 5$

vertical line



8. (18pts) A parametric curve is given by $x(t) = 2t^3 - 3t^2$, $y(t) = t^2 - 6t + 2$.
a) Find the points on the curve where the tangent line is horizontal or vertical.
b) Where does the curve go as $t \to \infty$ and $t \to -\infty$? (That is, find $\lim_{t\to\pm\infty} x(t)$, $\lim_{t\to\pm\infty} y(t)$.)
c) Plot the points from a) on a coordinate system and use them, along with information from b), or from plotting additional points, to get a graph of the curve. Recall that the curve moves in only one of general directions ↗ ↖ ↙ ↘ between points from a).

a) $x'(t) = 6t^2 - 6t$

$y'(t) = 2t - 6$

$6t^2 - 6t = 0 \qquad 2t - 6 = 0$

$6t(t-1) = 0 \qquad t = 3$

$t = 0, 1$

t	x	y
0	0	2
1	-1	-3
3	27	-7

$y(3) = 2\cdot 27 - 3\cdot 9 = 27$

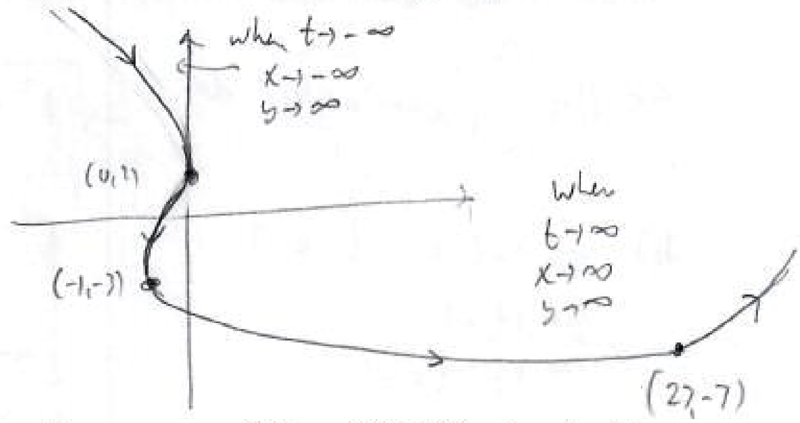
b) $\lim_{t\to\pm\infty} x(t) = \lim_{x\to\pm\infty} 2t^3 - 3t^2$

$= \lim_{x\to\pm\infty} t^3\left(2 - \underbrace{\frac{3}{t}}_{\to 0}\right) = \pm\infty \cdot 2$

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(t^2 - 6t + 2)$

$= \lim_{t\to\infty} t^2\left(1 - \underbrace{\frac{6}{t}}_{\to 0} + \underbrace{\frac{2}{t^2}}_{\to 0}\right) = \infty \cdot 1$

c)



Bonus. (10pts) Show that the length of a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ is given by the formula below. *Hint: to start, get a parametrization $(x(\theta), y(\theta))$ of the polar curve by using formulas for the polar → cartesian conversion.*

$$l = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

$\int_\alpha^\beta \sqrt{\left(\frac{d}{d\theta} f(\theta)\cos\theta\right)^2 + \left(\frac{d}{d\theta} f(\theta)\sin\theta\right)^2}\, d\theta$

$x = r\cos\theta = f(\theta)\cos\theta$

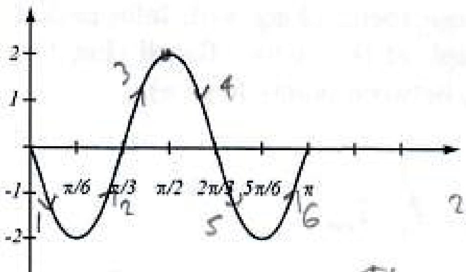
$y = r\sin\theta = f(\theta)\sin\theta$

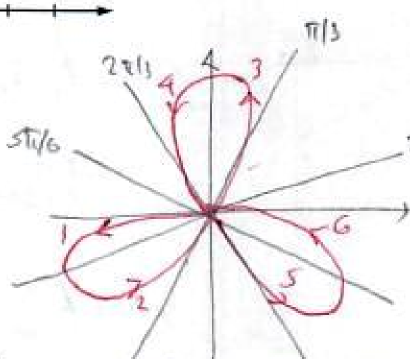
$= \int_\alpha^\beta \sqrt{\left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2}$

$= \int_\alpha^\beta \sqrt{f'(\theta)^2\cos^2\theta - 2f(\theta)\cancel{f'(\theta)\cos\theta\sin\theta} + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + 2f(\theta)\cancel{f'(\theta)\sin\theta\cos\theta} + f(\theta)^2\cos^2\theta}$

$= \int_\alpha^\beta \sqrt{f'(\theta)^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + f(\theta)^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

6. (12pts) The graph of $r = f(\theta)$ is given in cartesian coordinates. Use its intervals of increase and decrease to help you sketch the polar curve $r = f(\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.





(figure)

7. (16pts) The parametric curve $x = 3 + t$, $y = 1 + 2t$, $-1 \le t \le 2$ is given.

a) Use an integral to find the length of the curve.

b) Eliminate the parameter in order to find out what the curve is.

c) Use a college algebra method to find its length and compare your answer to a).

a) $l = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_{-1}^{2} \sqrt{1^2 + 2^2}\,dt = \int_{-1}^{2} \sqrt{5}\,dt = \sqrt{5}(2-(-1)) = 3\sqrt{5}$

b) $x = 3+t \leftarrow t = x-3$

$y = 1+2t$

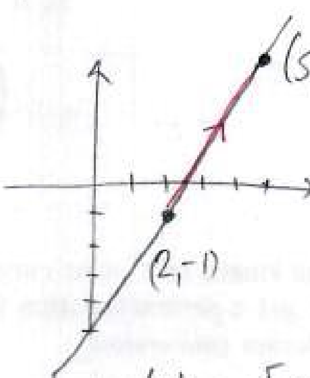
$= 1+2(x-3)$

$y = 2x-5$

Part of line $y = 2x-5$

When t goes from -1 to 2

$x = 3+t$ goes from 2 to 5



c) distance from $(2,-1)$ to $(5,5)$

is $\sqrt{(5-2)^2 + (5-(-1))^2} = \sqrt{9+36}$

$= \sqrt{45} = 3\sqrt{5}$, Same