Calculus 2 — Exam 4
MAT 308, Fall 2021 — D. Ivanšić

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Show all your work!

Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (16pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n^2} \cdot (x-1)^n$

$$\sqrt[n]{\left|\frac{1}{2^n n^2}(x-1)^n\right|} = \sqrt[n]{\frac{1}{2^n n^2}|x-1|^n} = \frac{1}{2\sqrt[n]{n^2}}|x-1| \to \frac{|x-1|}{2}$$

$\sqrt[n]{n^2} \to 1$

$$\frac{|x-1|}{2} < 1$$

$$|x-1| < 2$$

-2 $+2$

-1 1 3

$x=3$ gives $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n^2}(3-1)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges p-series, $p>1$

2^n

$x=-1$ gives $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n^2}(-1-1)^n = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^n n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$

$(-2)^n$

Converges absolutely, because $\sum \frac{1}{n^2}$ converges

Interval: $[-1, 3]$

2. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)} x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{1\cdot 3\cdot 5 \cdots (2n-1)(2(n+1)-1)}\cancel{x^{n+1}}}{\frac{1}{1\cdot 3\cdot 5 \cdots (2n-1)}\cancel{x^n}}\right| = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)} \cdot |x|$$

x

$$= \frac{|x|}{2n+1} \to \frac{|x|}{\infty} = 0,$$

$0<1$ so series converges for every x

Interval: $(-\infty, \infty)$

3. (6pts) Use a known power series to find the sum. It's not a typo — it really is $(2n)!$ in the denominator, not $(2n+1)!$ Think.

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{3^{2n+1}(2n)!} = \frac{\pi}{3}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{3^{2n}(2n)!} = \frac{\pi}{3}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \frac{\pi}{3}\cdot\cos\frac{\pi}{3}$$

$$= \frac{\pi}{6}$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

4. (8pts) Use a known power series to find the limit.

$$\lim_{x\to 0}\frac{\ln(1+3x^3)-3x^3}{x^6} = \lim_{x\to 0}\frac{\cancel{3x^3}-\frac{(3x^3)^2}{2}+\frac{(3x^3)^3}{3}-\dots\;\cancel{-3x^3}}{x^6}$$

$$= \lim_{x\to 0}\frac{-\frac{9x^6}{2}+\frac{27x^9}{3}-\dots}{x^6} = \lim_{x\to 0}\frac{\cancel{x^6}\left(-\frac{9}{2}+\frac{27x^3}{3}-\dots\right)}{\cancel{x^6}}$$

$$= -\frac{9}{2}+0+0\dots = -\frac{9}{2}$$

5. (14pts) Use geometric series to get a power series for $\dfrac{2x-8}{x^2-8x+15}$. The partial fraction decomposition has been written for you. Your answer needs to be a single sum of type $\sum c_n x^n$. State the interval of convergence (no need to check the endpoints).

$$\frac{2x-8}{x^2-8x+15} = \frac{1}{x-5}+\frac{1}{x-3} = -\frac{1}{5-x}-\frac{1}{3-x} = -\frac{1}{5(1-\frac{x}{5})}-\frac{1}{3(1-\frac{x}{3})}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}-\frac{x^n}{5^{n+1}}+\sum_{n=0}^{\infty}-\frac{x^n}{3^n}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{5^{n+1}}-\frac{1}{3^{n+1}}\right)x^n$$

For convergence, we need:

$\left|\frac{x}{5}\right|<1$ and $\left|\frac{x}{3}\right|<1$

$|x|<5$ and $|x|<3$

overlap is $|x|<3$

so $-3<x<3$, interval is $(-3,3)$

6. (12pts) Use a geometric series and antidifferentiation to find the McLaurin series for $\arctan x$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \Big/ \int$$

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$x=0$ gives $0 = C$, so $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

7. (18pts) Let $f(x) = \ln x$.
a) Find the 3nd Taylor polynomial for f centered at $a = 4$.
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $[3, 5]$. Leave your answer as a fraction.

a)

$y = \ln x$	$y(4) = \ln 4$
$y' = \frac{1}{x} = x^{-1}$	$y'(4) = \frac{1}{4}$
$y'' = -x^{-2}$	$y''(4) = -\frac{1}{16}$
$y''' = 2x^{-3}$	$y'''(4) = \frac{2}{64} = \frac{1}{32}$
$y^{(4)} = -6x^{-4}$	

$$T_3(x) = \ln 4 + \frac{1}{4}(x-1) - \frac{\frac{1}{16}}{2!}(x-4)^2 + \frac{\frac{1}{32}}{3!}(x-4)^3$$

$$= \ln 4 + \frac{1}{4}(x-1) - \frac{1}{32}(x-4)^2 + \frac{1}{192}(x4)^3$$

$$|f^{(4)}(x)| = \frac{6}{x^4}$$ max at 3, value $= \frac{6}{3^4}$

3 4 5

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-4)^4$$

$$|R_3(x)| = \frac{|f^{(4)}(z)|}{4!}\underbrace{|x-4|^5}_{\le 1} \le \frac{\frac{6}{3^4}}{4!}\cdot 1^5 = \frac{\cancel{6}}{3^4\cdot 1\cdot \cancel{2\cdot 3}\cdot 4} = \frac{1}{81\cdot 4} = \frac{1}{324}$$

8. (16pts) Use the known power series for $\sin x$ to find the series representing $\int_0^1 \sin(x^2)\,dx$. (Note that $\sin(x^2)$ does not have an antiderivative that is an elementary function.) Give an approximation of the definite integral with accuracy 10^{-4}. Write the approximation as a sum (you do not have to simplify it).

$$\int_0^1 \sin x^2\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}\Bigg|_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+3)(2n+1)!}$$

denom. increasing, so terms are decreasing

alt. series est:

$|s - s_n| \le b_{n+1}$

Need $(4n+3)(2n+1)! \geq 10^4$

n	$(4n+3)(2n+1)!$
2	$11 \cdot 5! = 11 \cdot 120 < 10^4$
3	$15 \cdot 7! = 15 \cdot 5040 > 10^4$

$$s_2 = \frac{1}{3 \cdot 1} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!}$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320}$$

approximates integral with accu. 10^{-4}

Bonus (10pts) Find a fraction that is the approximation of e with accuracy 10^{-3}. Use the series for e^x and Taylor's formula, and assume you know $e < 3$.

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$|R_n(x)| \le \left| \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} \right|$$

since $0 \le z \le 1$
$1 \le e^z \le e \le 3$

$$|R_n(1)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}(1-0)^{n+1} = \frac{|e^z|}{(n+1)!} \le \frac{3}{(n+1)!}$$

When $n+1 = 7$ get desired accu., so use $n = 6$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

Need $\frac{3}{n!} \le 10^{-3}$

$\frac{n!}{3} \geq 10^3$

n	$\frac{n!}{3}$
5	$\frac{5!}{3} = 40$ 2.6
6	$\frac{6!}{3} = 240$ 2.7
7	$\frac{7!}{3} = 1680 \geq 10^3$