Calculus 2 — Exam 3
MAT 308, Fall 2021 — D. Ivanšić

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Show all your work!

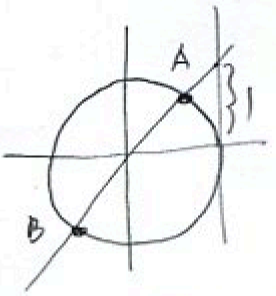
Find the limits, if they exist.

1. (6pts) $\lim_{n\to\infty} (-1)^n \frac{2^{2n+3}}{5^n} = \lim_{n\to\infty} (-1)^n \frac{2^{2n}\cdot 2^3}{5^n} = \lim_{n\to\infty} \left(-\frac{2^2}{5}\right)^n 8 = \lim_{n\to\infty} \left(-\frac{4}{5}\right)^n \cdot 8$

$= 0\cdot 8 = 0$

$(r^n \to 0 \text{ if } |r|<1)$

2. (6pts) $\lim_{n\to\infty} \tan \frac{(4n+1)\pi}{4} = \lim_{n\to\infty} 1 = 1$



tan of $\underbrace{\frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \dots}_{}$ $= \left(\frac{1}{4}+k\right)\pi = \frac{\pi}{4}+n\pi$

all angles corresponding to points A, B on unit circle have tangent 1.

3. (10pts) Find the limit. Use the theorem that rhymes with the blank from "______ Navidad."

$\lim_{n\to\infty} \frac{5+(-1)^n}{n^4+3n^2}$

$-1 \le (-1)^n \le 1$

$5-1 \le 5+(-1)^n \le 5+1 \quad | \div n^4+3n^2$

$\frac{4}{n^4+3n^2} \le \frac{5+(-1)^n}{n^4+3n^2} \le \frac{6}{n^4+3n^2}$

$\left.\begin{aligned} \lim_{n\to\infty} \frac{4}{n^4+3n^2} &= \frac{4}{\infty} = 0 \\ \lim_{n\to\infty} \frac{6}{n^4+3n^2} &= \frac{6}{\infty} = 0 \end{aligned}\right\}$

By the squeeze theorem,

$\lim_{n\to\infty} \frac{5+(-1)^n}{n^4+3n^2} = 0$

4. (6pts) Write the series using sigma notation:

$$\frac{7}{5}-\frac{9}{25}+\frac{11}{125}-\frac{13}{625}+\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{7+2(n-1)}{5^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{5+2n}{5^n}$$

$7 \quad 7+2 \quad 7+4 \quad 7+6$

$5^1 \quad 5^2 \quad 5^3 \quad 5^4$

5. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty}\frac{2^{3n}}{5\cdot 3^{2n+3}} = \sum_{n=1}^{\infty}\frac{2^{3n}}{5\cdot 3^{2n}\cdot 3^3} = \sum_{n=1}^{\infty}\frac{(2^3)^n}{5\cdot 3^3\cdot(3^2)^n} = \sum_{n=1}^{\infty}\frac{1}{5\cdot 3^3}\left(\frac{2^3}{3^2}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{5\cdot 3^3}\left(\frac{8}{9}\right)^n = \frac{\frac{2^3}{5\cdot 3^5}}{1-\frac{8}{9}} = \frac{8}{5\cdot \cancel{3^5}_{3^3}}\cdot\frac{\cancel{9}^{1}}{1} = \frac{8}{5\cdot 27} = \frac{8}{135}$$

$\left|\frac{8}{9}\right|<1$ so geom. series converges

Determine whether the following series converge and justify your answer.

6. (12pts) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{n^2+5n-3}$

$$\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

$$\frac{\frac{\sqrt{n}+1}{n^2+5n-3}}{\frac{1}{n^{3/2}}} = \frac{\frac{\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)}{n^2\left(1+\frac{5}{n}-\frac{3}{n^2}\right)}}{\frac{1}{n^{3/2}}} = \frac{\cancel{\frac{1}{n^{3/2}}}\,\frac{1+\frac{1}{\sqrt{n}}}{1+\frac{5}{n}-\frac{3}{n^2}}}{\cancel{\frac{1}{n^{3/2}}}} \to \frac{1+0}{1+0-0} = 1$$

Since $\sum\frac{1}{n^{3/2}}$ converges (p-series, $p>1$), $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{n^2+5n-3}$ converges by limit comparison test

7. (6pts) $\displaystyle\sum_{n=1}^{\infty}\left(\sin\frac{1}{n}-\sqrt[n]{2}\right)$

$$\lim_{n\to\infty}\left(\sin\frac{1}{n}-\sqrt[n]{2}\right) = \sin\frac{1}{\infty}-1 = \sin 0 - 1 = -1 \neq 0$$

diverges by divergence test

8. (8pts) Consider the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$.

a) Is the series convergent? Justify.
b) Is the series absolutely convergent? Justify.

a) $\left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing (denominator increasing)

$\frac{1}{\sqrt{n}} \to \frac{1}{\infty} = 0$

By alternating series test,
series converges

b) Consider $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$

This series diverges.
p-series, $p = \frac{1}{2} < 1$,

9. (12pts) For the sequence $\left\{\frac{n}{(n+5)^2}\right\}_{n=1}^{\infty}$, determine:

a) for which n the sequence is decreasing.
b) its upper and lower bounds.

a) $f(x) = \frac{x}{(x+5)^2}$ $\quad f'(x) = \frac{1\cdot(x+5)^2 - x\cdot 2(x+5)}{(x+5)^4} = \frac{(x+5)(x+5-2x)}{(x+5)^4} = \frac{5-x}{(x+5)^3}$

Since $x > 0$, $(x+5)^3 > 0$

$5-x$: $+$ (0) $-$ at 5

$f'(x) < 0$ for $x > 5$. Sequence decreases for $n \geq 5$

b) $0 \leq \frac{n}{(n+5)^2}$ $\quad n \leq n+5 \leq (n+5)^2$
clearly $\quad$ So $\frac{n}{(n+5)^2} \leq 1$

Determine whether the following series converge using the root or ratio test.

10. (11pts) $\sum_{n=2}^{\infty}\frac{2^{2n}(n^2+n+7)}{3^{n+1}(n^3-n^2)}$

$$\sqrt[n]{\left|\frac{2^{2n}(n^2+n+7)}{\underbrace{3^{n+1}(n^3-n^2)}_{\text{pos for } n\geq 2}}\right|} = \sqrt[n]{\frac{(2^2)^n(n^2+n+7)}{3^n\cdot 3(n^3-n^2)}} = \frac{\sqrt[n]{(2^2)^n}\,\sqrt[n]{n^2+n+7}}{\sqrt[n]{3^n}\,\sqrt[n]{3}\,\sqrt[n]{n^3-n^2}} \to \frac{2^2\cdot 1}{3\cdot 1\cdot 1} = \frac{4}{3} > 1$$

Series diverges by root test

11. (11pts) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^{11}}{4^n \cdot n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n \frac{(n+1)^{11}}{4^{n+1}(n+1)!}}{(-1)^{n-1}\frac{n^{11}}{4^n \cdot n!}}\right| = \frac{(n+1)^{11}}{\underset{\cancel{4^n}\cdot 4^1}{4^{n+1}}(n+1)\cancel{n!}} \cdot \frac{\cancel{4^n \cdot n!}}{n^{11}}$$

$$= \left(\frac{n+1}{n}\right)^{11} \cdot \frac{1}{4\cdot(n+1)} = \left(1+\frac{1}{n}\right)^{11} \cdot \frac{1}{4(n+1)} \to (1+0)^{11} \cdot \frac{1}{4\cdot\infty} = \frac{1}{\infty} = 0 < 1$$

Series converges absolutely by ratio test.

Bonus. (10pts) Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$below.

a) Show that $\lim_{n\to\infty} b_n = 0$ by considering the odd and even terms separately.
b) Show that the sequence b_n is not decreasing.
c) Show that the partial sum s_{2n} of the series below satisfies $s_{2n} = u_n - v_n$, where u_n, v_n are partial sums of familiar series.
d) Use c) to help you answer: does the series below converge?

$$\frac{1}{1} - \frac{1}{2^1} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{2^3} + \frac{1}{4} - \frac{1}{2^4} + \cdots =$$

a) odd terms $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\} \to 0$ $\left(\lim_{n\to\infty}\frac{1}{n}=0\right)$

even terms $\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots\} \to 0$ $\left(\lim_{n\to\infty}\frac{1}{2^n} = \frac{1}{\infty} = 0\right)$

b) $\{b_n\} = 1 > \frac{1}{2^1} < \frac{1}{2} > \frac{1}{2^2} < \frac{1}{3} > \frac{1}{2^3} < \frac{1}{4} > \frac{1}{2^4}$ $\qquad \frac{1}{n} > \frac{1}{2^n} < \frac{1}{n+1}$ in general

c) $s_{2n} = \frac{1}{1} - \frac{1}{2^1} + \frac{1}{2} - \frac{1}{2^2} + \dots + \frac{1}{n} - \frac{1}{2^n} = \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n}}_{u_n} - \underbrace{\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right)}_{v_n}$

p-series, p=1 $\qquad$ geom series

d) Since $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} u_n - \lim_{n\to\infty} v_n = \infty - \frac{\frac{1}{2}}{1-\frac{1}{2}} = \infty - 1 = \infty$,

series diverges.