Calculus 2 — Exam 2
MAT 308, Fall 2021 — D. Ivanšić

Name: Saul Ocean

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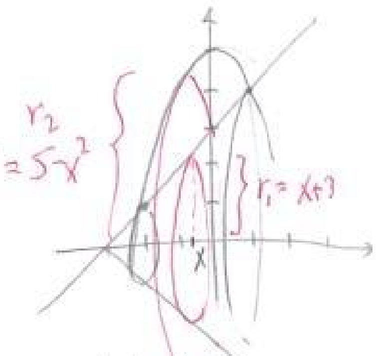
1. (24pts) The region bounded by the curves $y = 5 - x^2$ and $y = x + 3$ is rotated around the x-axis.
a) Sketch the solid and a typical cross-sectional washer.
b) Set up the integral for the volume of the solid.
c) Evaluate the integral.



Area of cross-section: $\pi r_2^2 - \pi r_1^2 = \pi\left((5-x^2)^2-(x+3)^2\right)$

$$V = \int_{-2}^{1} \pi\left((5-x^2)^2-(x+3)^2\right)dx = \pi\int_{-2}^{1} 25-10x^2+x^4-(x^2+6x+9)\,dx$$

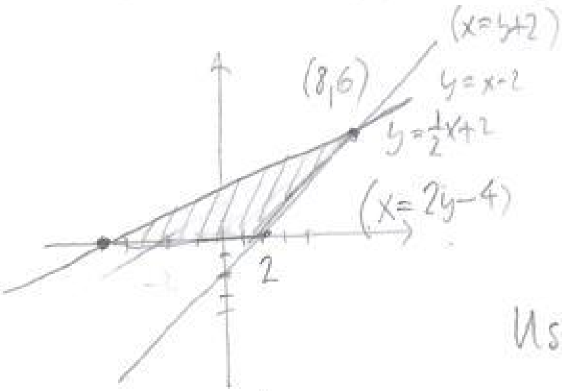
$$= \pi\int_{-2}^{1} x^4-11x^2-6x+16\,dx = \pi\left(\left(\frac{x^5}{5}-\frac{11x^3}{3}-3x^2\right)\Big|_{-2}^{1}+16(1-(-2))\right)$$

$$= \pi\left(\frac{1}{5}(1-(-32)) - \frac{11}{\cancel{5}\,3}(1-(-8)) - 3(1-4) + 48\right)$$

$$= \pi\left(\frac{33}{5}-33+9+48\right) = \pi\left(\frac{33}{5}+24\right) = \frac{153\pi}{5}$$

Intersection:
$5-x^2 = x+3$
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x=-2,1$

2. (14pts) Consider the triangle bounded by the x-axis and lines $y = \frac{1}{2}x + 2$ and $y = x - 2$.
a) Sketch the triangle.
b) Set up the integral that computes its area. Simplify, but do not evaluate the integral.



$$A = \int_0^6 (y+2)-(2y-4)\,dy = \int_0^6 6-y\,dy$$

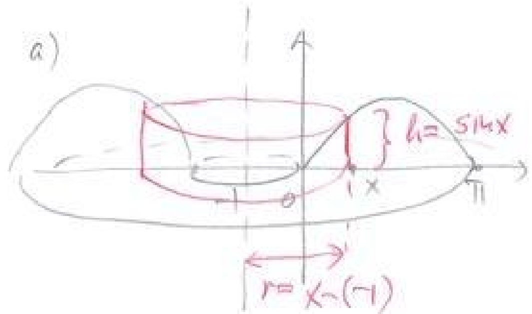
Using x: $A = A_1 + A_2 = \int_{-4}^{2} \frac{1}{2}x+2-0\,dx + \int_{2}^{8} \frac{1}{2}x+2-(x-2)\,dx$

$$= \int_{-4}^{2} \frac{1}{2}x+2\,dx + \int_{2}^{8} 4-\frac{1}{2}x\,dx$$

$x-2 = \frac{1}{2}x+2$
$x-\frac{1}{2}x = 4$
$\frac{1}{2}x = 4$
$x=8$

3. (16pts) Rotate the region bounded by the curve $y = \sin x$ for $0 \le x \le \pi$ and the x-axis about the line $x = -1$ to get a solid.
a) Sketch the solid and a typical cylindrical shell.
b) Set up the integral for the volume of the solid using the shell method. Simplify, but do not evaluate the integral.

a)

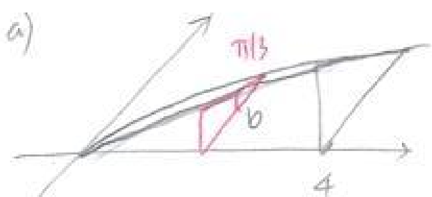


b) $V = \int_0^\pi 2\pi r h \, dx$

$= \int_0^\pi 2\pi (x+1) \sin x \, dx$

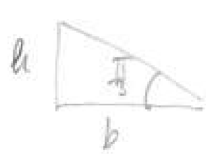
4. (18pts) The base of a solid is the region in the xy-plane bounded by curves $y = \sqrt{x}$, $x = 4$ and the x-axis. The cross-sections of the solid perpendicular to the x-axis are right triangles, where one side lies on the base, the other is perpendicular to the base, and the hypothenuse makes angle $\frac{\pi}{6}$ with the side lying in the base.
a) Sketch the solid and a typical cross-section.
b) Set up the integral for the volume of the solid.
c) Evaluate the integral.

a)



b) $V = \int_0^4 A(x)\, dx = \int_0^4 \frac{x}{8\sqrt{3}}\, dx$

c) $= \frac{1}{2\sqrt{3}} \cdot \frac{x^2}{2}\Big|_0^4 = \frac{1}{4\sqrt{3}}(16-0) = \frac{4}{\sqrt{3}}$



$\frac{h}{b} = \tan\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

$h = \frac{b}{\sqrt{3}}$

$A = \frac{1}{2} b h = \frac{1}{2} \frac{b}{\sqrt{3}} \cdot b = \frac{b^2}{2\sqrt{3}} = \frac{\sqrt{x}^2}{2\sqrt{3}} = \frac{x}{2\sqrt{3}}$

5. (12pts) Set up the integral for the length of the curve $y = e^x - x$ from $x = 0$ to $x = 3$. Simplify, but do not evaluate the integral.

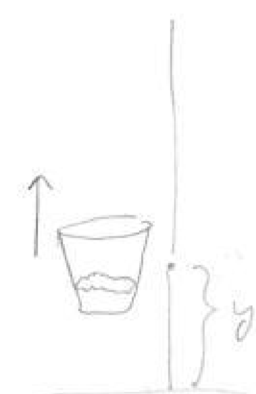
$$l = \int_0^3 \sqrt{1+((e^x-x)')^2}\, dx = \int_0^3 \sqrt{1+(e^x-1)^2}\, dx = \int_0^3 \sqrt{1+e^{2x}-2e^x+1}\, dx$$

$$= \int_0^3 \sqrt{e^{2x}-2e^x+2}\, dx$$

6. (16pts) On a building site in heavy rain, a bucket weighing 50kg is lifted 60 meters. The amount of water in the bucket is 100 liters at bottom and 120 liters by the time it reaches the top. Set up the integral for the work needed to lift the bucket. (Ignore cable weight.) Assume $g = 10$ and water density $= 1$kg/liter. Simplify, but do not evaluate the integral.

water gets added at rate:

$\frac{120-100}{60} = \frac{20}{60} = \frac{1}{3}$ liters per meter traveled



density

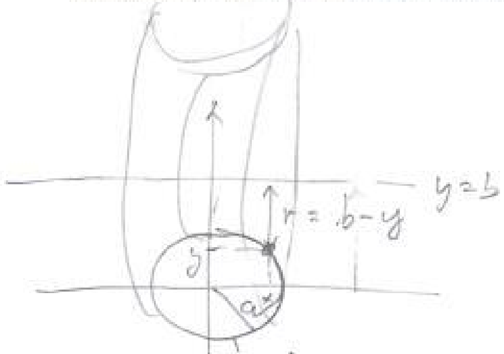
mass at height $y = 50 + (100 + \frac{1}{3}y) \cdot 1 = 150 + \frac{1}{3}y$

$$W = \int_0^{60} F(y)\, dy = \int_0^{60} \left(150 + \frac{1}{3}y\right) \cdot 10\, dy$$

g

$$= \int_0^{60} 1500 + \frac{10}{3}y\, dy$$

Bonus (10pts) Let $0 < a < b$. By rotating the circle of radius a centered at the origin around the line $y = b$, you will get a torus (surface of a doughnut). Set up the integral for the surface area of the torus. (Tip: exploit symmetry about the y-axis for a simpler solution.) Do not evaluate the integral.



Surface area of torus = 2 · surface area of right half,

$$= 2\cdot\int 2\pi r\,ds = 4\pi\int_{-a}^{a}(b-y)\sqrt{1+\left(-\frac{y}{\sqrt{a^2-y^2}}\right)^2}\,dy$$

$$= 4\pi\int_{-a}^{a}(b-y)\sqrt{1+\frac{y^2}{a^2-y^2}}\,dy$$

$$= 4\pi\int_{-a}^{a}(b-y)\sqrt{\frac{a^2-\cancel{y^2}+\cancel{y^2}}{a^2-y^2}}\,dy$$

$$= 4\pi\int_{-a}^{a}\frac{a(b-y)}{\sqrt{a^2-y^2}}\,dy$$

$x^2+y^2=a^2$

$x=\sqrt{a^2-y^2}$ right half

$x' = \frac{1}{2\sqrt{a^2-y^2}}\cdot(-2y) = -\frac{y}{\sqrt{a^2-y^2}}$

Using x requires two integrals

$y=\pm\sqrt{a^2-x^2}$

$y'=\mp\frac{x}{\sqrt{a^2-x^2}}$

$$\text{Surface area} = 2\pi\int_{-a}^{a}\left(b-\sqrt{a^2-x^2}\right)\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$+\ 2\pi\int_{-a}^{a}\left(b+\sqrt{a^2-x^2}\right)\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$= 2\pi\left(\int_{-a}^{a}\left(b-\sqrt{a^2-x^2}\right)\frac{a}{\sqrt{a^2-x^2}} + \int_{-a}^{a}\left(b+\sqrt{a^2-x^2}\right)\frac{a}{\sqrt{a^2-x^2}}\,dx\right)$$

$$= \int_{-a}^{a}\frac{4\pi ab}{\sqrt{a^2-x^2}}\,dx$$