Calculus 2 — Exam 1 MAT 308, Fall 2021 — D. Ivanšić

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Find the following integrals:

1. (12pts) $\int e^x \cos x\,dx = \left[\begin{matrix} u=\cos x & dv=e^x\,dx \\ du=-\sin x\,dx & v=e^x \end{matrix}\right] = e^x\cos x + \int e^x \sin x\,dx$

$= \left[\begin{matrix} u=\sin x & dv=e^x\,dx \\ du=\cos x\,dx & v=e^x, \end{matrix}\right] = e^x\cos x + e^x \sin x - \int e^x \cos x$

$\Rightarrow \quad 2\int e^x \cos x\,dx = e^x\cos x + e^x \sin x$

$\int e^x \cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x) + C$

2. (8pts) $\int \sin^2 x \cos^2 x\,dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}\,dx$

$= \frac{1}{4}\int 1-\cos^2(2x)\,dx = \frac{1}{4}\int \sin^2(2x)\,dx$

$= \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx = \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C$

Determine whether the following improper integral converges by calculating it directly.

3. (10pts) $\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{\arctan x}{1+x^2}\,dx = \left[\begin{matrix} w=\arctan x & x=t,\ w=\arctan t \\ dw=\frac{1}{1+x^2}\,dx & x=0,\ w=\arctan 0=0 \end{matrix}\right]$

$= \lim_{t\to\infty}\int_0^{\arctan t} u\,du = \lim_{t\to\infty} \left.\frac{u^2}{2}\right|_0^{\arctan t} = \lim_{t\to\infty}\frac{1}{2}(\underbrace{\arctan^2 t}_{\to\frac{\pi}{2}} - 0)$

$= \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$

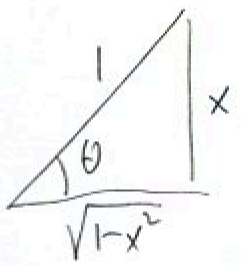
Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (14pts) $\int x^3\sqrt{1-x^2}\,dx = \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} = \int \sin^3\theta\sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$

$$= \int \sin^3\theta\cos^2\theta\, d\theta = \int \cos^2\theta\,\underbrace{\sin^2\theta}_{1-\cos^2\theta}\,\sin\theta\, d\theta$$

$$= \begin{bmatrix} w = \cos\theta \\ dw = -\sin\theta\, d\theta \end{bmatrix} = \int u^2(1-u^2)(-du)$$

$$= \int u^4 - u^2\, du = \frac{u^5}{5} - \frac{u^3}{3} = \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}$$



$$= \frac{(1-x^2)^{\frac{5}{2}}}{5} - \frac{(1-x^2)^{\frac{3}{2}}}{3} + C$$

5. (14pts) $\int_0^3 \frac{1}{(9+x^2)^{\frac{3}{2}}}\,dx = \begin{bmatrix} x = 3\tan\theta & 3 = 3\tan\theta,\ \tan\theta = 1,\ \theta = \frac{\pi}{4} \\ dx = 3\sec^2\theta\, d\theta & 0 = 3\tan\theta\ \tan\theta = 0,\ \theta = 0 \end{bmatrix}$

$$= \int_0^{\pi/4} \frac{1}{(9+9\tan^2\theta)^{3/2}} \cdot 3\sec^2\theta\, d\theta = \int_0^{\pi/4} \frac{3\cdot\sec^2\theta\, d\theta}{9^{3/2}(\sec^2\theta)^{3/2}}$$

$9(1+\tan^2\theta) = 9\sec^2\theta$

$$= \int_0^{\pi/4} \frac{3\sec^2\theta}{27\sec^3\theta}\, d\theta = \frac{1}{9}\int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int_0^{\pi/4} \cos\theta\, d\theta$$

$$= \frac{1}{9}\sin\theta\Big|_0^{\pi/4} = \frac{1}{9}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{\sqrt{2}}{18}$$

Use the method of partial fractions to find the integral.

6. (14pts) $\int \frac{-x^2+2x+3}{(x-1)^3}\,dx = \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$ $\quad | \cdot (x-1)^3$

$-x^2+2x+3 = A(x-1)^2 + B(x-1) + C$

$-x^2+2x+3 = A(x^2-2x+1) + B(x-1) + C$

$x^2 \quad -1 = A \quad \Rightarrow A=-1$

$x^1 \quad 2 = -2A+B \quad \Rightarrow B=0$

$x^0 \quad 3 = A-B+C \Rightarrow 3=-1+C, \; C=4$

$= \int \frac{-1}{x-1} + \frac{4}{(x-1)^3}\,dx = -\ln|x-1| + 4\frac{(x-1)^{-2}}{-2}$

$(x-1)^{-3} \quad = -\ln|x-1| - \frac{2}{(x-1)^2} + C$

7. (10pts) Use comparison to determine whether the improper integral $\int_2^\infty \frac{x^5}{x^6-4}\,dx$ converges.

$\frac{x^5}{x^6-4} > \frac{x^5}{x^6} = \frac{1}{x}$

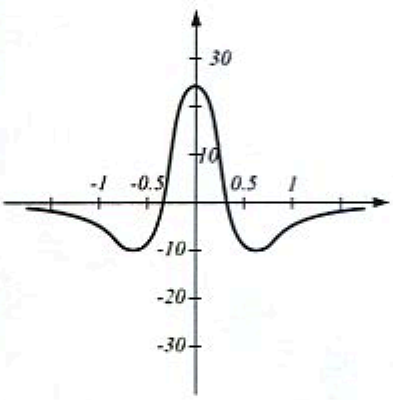
bigger denom.

Since $\int_2^\infty \frac{1}{x}\,dx$ diverges,

so does $\int_2^\infty \frac{x^5}{x^6-4}\,dx$

by comparison theorem

8. (18pts) Suppose we wanted to approximate the number $\frac{\pi}{4} = \arctan 1$. We could do it by approximating the integral $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$, which uses only the four algebraic operations.
a) Write the expression you would use to calculate S_6, the Simpson rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) Find the error estimate for S_n in general. The graph of the fourth derivative of $\frac{1}{1+x^2}$, which is $\frac{24(5x^4-10x^2+1)}{(1+x^2)^5}$, is shown in the picture.
c) Estimate the error for S_6.
d) What should n be in order for S_n to give you an error less than 10^{-8}?



a) Δx

$0 \quad \frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{5}{6} \quad 1$

$$S_6 = \frac{\frac{1}{6}}{3}\left(\frac{1}{1+0^2} + \frac{4}{1+\left(\frac{1}{6}\right)^2} + \frac{2}{1+\left(\frac{1}{3}\right)^2} + \frac{4}{1+\left(\frac{1}{2}\right)^2} + \frac{2}{1+\left(\frac{2}{3}\right)^2} + \frac{4}{1+\left(\frac{5}{6}\right)^2} + \frac{1}{1+1^2}\right)$$

b) $$|E_S| \le \frac{24\cdot(1-0)^5}{180\cdot n^4} = \frac{24}{180n^4} = \frac{2}{15n^4}$$

$|f^{(4)}(x)|$ has max at $x=0$

$f^{(4)}(0) = 24$

c) For $n=6$, get $$\frac{2}{15\cdot 6^4} = \frac{1}{15\cdot 6^3\cdot 3} = \frac{1}{9\cdot 5\cdot 216} = \frac{1}{9\cdot 1080} = \frac{1}{9720}$$

d) Wish to have $$\frac{2}{15n^4} \le 10^{-8} \qquad n \ge \sqrt[4]{\frac{2}{15}}\cdot 10^2$$

$$\frac{2}{15}\cdot 10^8 \le n^4 \qquad n \ge 100\sqrt[4]{\frac{2}{15}}$$

and even

Bonus (10pts) Determine for which $p > 0$ the integral below converges. (Note this is not the standard knowledge one because the interval is different.)

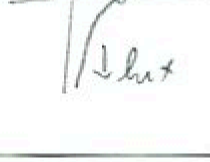
$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+}\int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to 0+} \left.\frac{x^{-p+1}}{-p+1}\right|_t^1 = \lim_{t\to 0+} \frac{1}{-p+1}\left(1-t^{-p+1}\right)$$

x^{-p}

$p \ne 1$

$$= \frac{1}{-p+1}\left(1-(0+)^{-p+1}\right) = \frac{1}{-p+1}\begin{cases} 1-0 & \text{if } -p+1>0,\ \text{i.e. } p<1 \\ 1-\infty & \text{if } -p+1<0,\ \text{i.e. } p>1 \end{cases}$$

For $p=1$: $$= \lim_{t\to 0+} \ln|x| \Big|_t^1 = \lim_{t\to 0+}(\ln 1 - \ln t) = -\ln 0+ = -(-\infty) = \infty$$



Converges for $p<1$

Diverges for $p \ge 1$