Calculus 2 — Exam 0
MAT 308, Fall 2021 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(3x^4 - e^6 + \sqrt[3]{x^5} + \frac{4}{x^7}\right) =$ $3\cdot 4x^3 - 0 + \frac{5}{3}x^{\frac{2}{3}} + 4\cdot(-7)x^{-8}$

$x^{\frac{5}{3}}$ $4x^{-7}$

$= 12x^3 + \frac{5}{3}x^{\frac{2}{3}} - \frac{28}{x^8}$

2. (6pts) $\frac{d}{dx}\left(e^{2x}\cos^3 x\right) =$ $e^{2x}\cdot 2\cdot\cos^3 x + e^{2x}\cdot 3\cos^2 x(-\sin x)$

$= e^{2x}\cos^2 x\,(2\cos x - 3\sin x)$

3. (8pts) $\frac{d}{dt}\frac{(t^2-1)^3}{(t^2+1)^2} =$ $\frac{3(t^2-1)^2(2t)(t^2+1)^2 - (t^2-1)^3 2(t^2+1)\cdot 2t}{(t^2+1)^4}$

$= \frac{(t^2-1)^2\cancel{(t^2+1)}\cdot 2t\left(3(t^2+1) - 2(t^2-1)\right)}{(t^2+1)^{\cancel{4}\,3}} = \frac{(t^2-1)^2\cdot 2t\,(t^2+5)}{(t^2+1)^3}$

4. (4pts) $\frac{d}{dx}\frac{1}{x\ln x} =$ $\frac{d}{dx}(x\ln x)^{-1} = -(x\ln x)^{-2}\cdot\left(1\cdot\ln x + x\cdot\frac{1}{x}\right)$

$= -\frac{\ln x + 1}{(x\ln x)^2}$

5. (6pts) $\frac{d}{d\theta}\arcsin^2(\tan\theta) =$ $2\arcsin(\tan\theta)\cdot\frac{1}{\sqrt{1-\tan^2\theta}}\cdot\sec^2\theta$

$= \frac{2\sec^2\theta\,\arcsin(\tan\theta)}{\sqrt{1-\tan^2\theta}}$

6. (7pts) Find the first and second derivatives of $f(x) = \cos(x^2)$.

$f'(x) = -\sin(x^2)\cdot 2x$

$f''(x) = -2\left(\cos(x^2)\cdot 2x\cdot x + \sin(x^2)\cdot 1\right)$

$= -2\left(2x^2\cos(x^2) + \sin(x^2)\right)$

7. (5pts) Let $f(x) = \dfrac{1}{x^3}$. Take the first four derivatives of f, and try to spot the pattern. What is $f^{(20)}(x)$, the 20th derivative of f? How about $f^{(n)}(x)$?

$y = x^{-3}$

$y' = -3x^{-4}$

exp. are derivative order + 3

$y'' = (-3)(-4)x^{-5}$

$y''' = (-3)(-4)(-5)x^{-6}$

$y^{(4)} = (-3)(-4)(-5)(-6)x^{-7}$

1 less than

$y^{(20)} = \overbrace{(-3)(-4)\ldots(-22)}^{\text{20 factors}}\, x^{-23}$

$= (-1)^{20}\, 3\cdot4\cdot\ldots\cdot22\, x^{-23}$

$= (-1)^{20}\dfrac{22!}{2}x^{-23}$

$y^{(n)} = (-1)^n \dfrac{(n+2)!}{2} x^{-(n+3)}$

Find the following limits. Use L'Hospital's rule if needed.

8. (2pts) $\lim\limits_{x\to0-}\dfrac{1}{x^5} = \dfrac{1}{0-} = -\infty$

9. (6pts) $\lim\limits_{x\to\infty}\dfrac{x^3-5x+4}{-3x^2+x+2} = \lim\limits_{x\to\infty}\dfrac{x^{\cancel{3}\,1}\left(1-\boxed{\frac{5}{x^2}}_{\to0}+\boxed{\frac{4}{x^3}}_{\to0}\right)}{\cancel{x^2}\left(-3+\boxed{\frac{1}{x}}_{\to0}+\boxed{\frac{2}{x^2}}_{\to0}\right)} = \lim\limits_{x\to\infty} x\cdot\dfrac{1-0+0}{-3+0+0}$

$= \infty\cdot\left(-\dfrac{1}{3}\right) = -\infty$

10. (8pts) $\lim\limits_{x\to\infty}\underbrace{(1+\ln x)^{\frac{1}{x}}}_{y} = \lim\limits_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^0 = 1$

$y = (1+\ln x)^{\frac{1}{x}}$

$\ln y = \ln(1+\ln x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln(1+\ln x) = \dfrac{\ln(1+\ln x)}{x}$

$\ln(1+\infty) = \infty$

$\lim\limits_{x\to\infty}\dfrac{\ln(1+\ln x)}{\underset{\to\infty}{x}} \overset{L'H}{=} \lim\limits_{x\to\infty}\dfrac{\frac{1}{1+\ln x}\cdot\frac{1}{x}}{1}$

$= \dfrac{1}{1+\infty}\cdot\dfrac{1}{\infty} = 0\cdot0 = 0$

Find the following antiderivatives.

11. (7pts) $\int 2x^4 - \frac{2}{1+x^2} + \sqrt[5]{x^6} + c^2\, dx =$ $\frac{2x^5}{5} - 2\arctan x + \frac{x^{\frac{11}{5}}}{\frac{11}{5}} + c^2 x$

$x^{\frac{6}{5}}$ ↑ constant for x

$= \frac{2x^5}{5} - 2\arctan x + \frac{5}{11} x^{\frac{11}{5}} + c^2 x + d$

since C is used for something else ↓

12. (3pts) $\int \sin\left(4x - \frac{\pi}{4}\right) dx = -\frac{\cos\left(4x - \frac{\pi}{4}\right)}{4}$

13. (7pts) $\int \frac{\sqrt{x} + \sqrt[3]{x}}{\sqrt[6]{x}}\, dx = \int \frac{x^{\frac{1}{2}} + x^{\frac{1}{3}}}{x^{1/6}}\, dx = \int x^{\frac{1}{2}-\frac{1}{6}} + x^{\frac{1}{3}-\frac{1}{6}}\, dx = \int x^{\frac{1}{3}} + x^{\frac{1}{6}}\, dx$

$= \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{7}{6}}}{\frac{7}{6}} = \frac{3}{4} x^{\frac{4}{3}} + \frac{6}{7} x^{\frac{7}{6}} + C$

Use the substitution rule in the following integrals:

14. (7pts) $\int \frac{e^x}{1+e^{2x}}\, dx =$ $\begin{bmatrix} u = e^x \\ du = e^x dx \end{bmatrix} = \int \frac{du}{1+u^2} = \arctan u$

$1+(e^x)^2$

$= 2\arctan e^x + C$

15. (10pts) $\int_{e^8}^{e^{64}} \frac{dx}{x\sqrt[3]{\ln x}} =$ $\begin{bmatrix} u = \ln x & x = e^{64} & u = 64 \\ du = \frac{1}{x} dx & x = e^8 & u = 8 \end{bmatrix}$

$= \int_8^{64} \frac{1}{\sqrt[3]{u}}\, du = \frac{3}{2} u^{\frac{2}{3}} \Big|_8^{64} = \frac{3}{2}\left(\sqrt[3]{64}^{\,2} - \sqrt[3]{8}^{\,2}\right)$

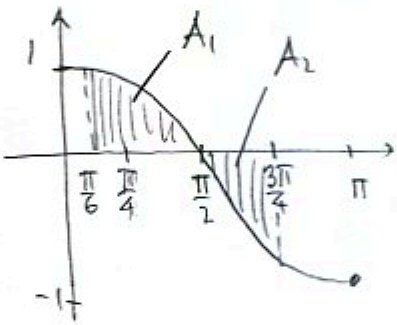
$u^{-1/3}$

$= \frac{3}{2}(16 - 4) = \frac{3}{\cancel{2}} \cdot \cancel{12}\ 6$

$= 18$

16. (8pts) Consider the integral $\int_{\frac{\pi}{6}}^{\frac{3\pi}{4}} \cos x \, dx$.

a) Draw a picture to explain the meaning of the integral.
b) Use the picture to estimate whether the integral is positive or negative.
c) Evaluate the integral to verify your finding in b).



a) $\int_{\pi/6}^{\frac{3\pi}{4}} \cos x \, dx = A_1 - A_2$

b) $A_1 - A_2 > 0$, since it appears $A_1 > A_2$

c) $\int_{\pi/6}^{\frac{3\pi}{4}} \cos x \, dx = \sin x \Big|_{\pi/6}^{\frac{3\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{1}{2} = \frac{\sqrt{2}-1}{2} > 0$

Bonus. (10pts) The rear inside cover of our book claims that

$$\int \frac{dx}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

Verify this formula by differentiating.

$$\frac{d}{dx} \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| = \frac{1}{2a} \frac{d}{dx} \left(\ln|x-a| - \ln|x+a| \right)$$

$$= \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) = \frac{1}{2a} \frac{x+a-(x-a)}{(x-a)(x+a)}$$

$$= \frac{2a}{2a(x^2-a^2)} = \frac{1}{x^2-a^2}$$