Calculus 2 — Lecture notes
MAT 308, Fall 2021 — D. Ivanšić

9.1 Parametric Curves

Problem: equations such as $y = f(x)$ cannot describe a curve that does not pass the vertical line test, like the one at right.

What we can do is view the curve as the path traced out by a moving particle. The position of a particle is known if we know its x- and y-coordinate at time t, in other words, if we are given how those coordinates vary with time. Thus, if we are given two functions $f(t)$ and $g(t)$ that give the x- and y-coordinates of the particle at time t, we know the position of the particle at time t.

$$x = f(t),\ y = g(t),\ a \le t \le b$$

The equations above are called *parametric equations* of the curve, and t is called the parameter (usually thought of as time).

Example. A curve is given by parametic equations $x = 1 + 3t$, $y = -1 - 2t$, $t \ge 0$. Draw the curve by eliminating the parameter, and taking into account the allowed values of the parameter.

Note. Eliminating the parameter, only gives us the equation in x, y of the curve that the parametric curve is **part of**. Further examination is needed to determine which part of the curve is described (traced out) by the parametrization.

Example. A curve is given by parametic equations $x = t^2 + 2t$, $y = t - 2$, t any real number.
a) Draw the curve by eliminating the parameter, and taking into account the allowed values of the parameter.
b) What part is traced out if $-2 \leq t \leq 2$?

Example. The equations $x = r \cos t$, $y = r \sin t$, $0 \le t \le 2\pi$ parametrize a circle of radius r, traced out counterclockwise, one revolution, starting at $(r, 0)$.

Write parametric equations that trace a circle of radius r in a clockwise way, starting at $(r, 0)$.

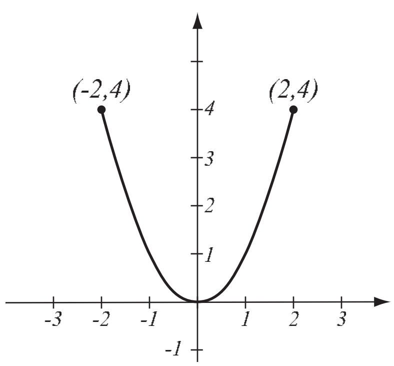
Write parametric equations that trace a circle of radius r in a clockwise way, starting at $(0, r)$.

Note: the same curve can be parametrized in many different ways.

Example. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is parametrized by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$.

Example. The graph of the function $y = f(x)$ is parametrized by $x = t$, $y = f(t)$, t in domain of f.

Example. Write parametric equations that describe the motion along the parabola $y = x^2$.



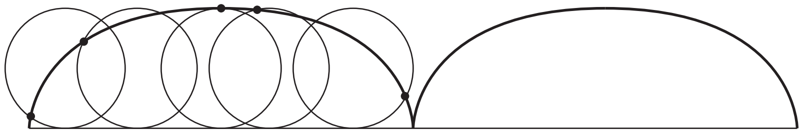
goes back and forth between
(-2,4) and *(2,4)* three times

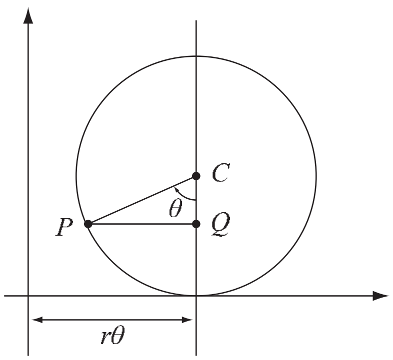
Example. Use a graphing device to graph curves given parametrically (t any real number):

a) $x = 4\cos t + \cos(4t)$
$y = 4\sin t - \sin(4t)$

b) $x = 4\cos(3t) + \cos(4t)$
$y = 4\sin(3t) - \sin(4t)$

Example. The curve traced out by a point on the circle as the circle rolls along a line is called a cycloid. Parametrize the cycloid.





9.2 Calculus with Parametric Curves

A *vector* is an arrow in the plane, used to describe quantities that have both a direction and a size, like the velocity of a moving particle, a force, etc. If the vector is placed in a coordinate system whose origin is at its tail and axes are parallel to an existing coordinate system, then the coordinates of the tip are referred to as the coordinates of the vector.

Suppose a particle is moving along a parametric curve (we think of the parameter t as time). At every moment, the particle has a *velocity* that is a vector tangent to the curve ("tangent vector"). If the parametrization of the curve is given by $x = x(t)$ and $y = y(t)$, the coordinates of the velocity vector are $(x'(t), y'(t))$.

The slope of the tangent line to the curve is $\dfrac{\text{rise}}{\text{run}}$ of the tangent vector, which is $\dfrac{y'(t)}{x'(t)}$.

Example. A curve is given by parametic equations $x = 1 - t^2$, $y = t^3 - t$, t any.
a) Find the tangent line to the curve at point $\left(\frac{3}{4}, -\frac{3}{8}\right)$.
b) Determine where on the curve the tangent line is horizontal or vertical.

Here is another way of looking at the formula for the slope of a tangent line, $\dfrac{y'(t)}{x'(t)}$. A parametric curve is not usually the graph of a function, but can be broken up into pieces that are graphs of functions.

More generally, let $G(x)$ be any function, and let $x = x(t)$. Then G is a function of t via x. If we know $\frac{d}{dt}G(x(t))$, how do we find $G'(x)$?

This helps us find the second derivative of F by x:

$$F''(x) = \frac{d}{dx}F'(x) = \frac{\frac{d}{dt}F'(x(t))}{x'(t)} = \frac{\frac{d}{dt}\frac{y'(t)}{x'(t)}}{x'(t)}$$

Example. Find intervals where the above curve, $x = 1 - t^2$, $y = t^3 - t$, is concave up and concave down.

Suppose part of a parametric curve $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$ is the graph of a function $F(x)$ between $a = x(\alpha)$ and $b = x(\beta)$. Then

$$\int_a^b F(x)\,dx = \int_\alpha^\beta y(t)x'(t)\,dt.$$

Example. Find the area of a disk of radius r using the parametric representation of a circle of radius r.

The length of a parametric curve $x = x(t)$, $y = y(t)$, $\alpha \le t \le \beta$, is given by

$$\text{length of curve} = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

assuming that the curve is traversed once for $\alpha \le t \le \beta$. Note that $\sqrt{x'(t)^2 + y'(t)^2}$ represents the length of the velocity vector.

Example. Find the circumference of a circle of radius r using its parametric representation.

Example. If we use the standard parametrization of a graph of a function we get the previous formula for length of curve.

9.3 Polar Coordinates

The position of a point in the plane is usually given by its *Cartesian coordinates* (x, y), where x and y represent directed distances from two perpendicular lines (axes).

Another way to specify the position of a point in the plane is to fix a point in the plane *(pole)*, and a ray emanating from this point *(polar axis)*, and give these two pieces of information:

$r =$ distance from point to the pole
$\theta =$ directed angle between polar axis and ray through point and pole

We extend polar coordinates to a negative r: if $r < 0$ the point lies on the ray opposite the ray making angle θ with the polar axis, and has distance $|r|$ from the pole.

Thus, polar coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point.

Note: in contrast with Cartesian coordinates, one point has infinitely many polar coordinates.

Example. Give some other polar coordinates for point with polar coordinates $\left(3, \frac{\pi}{2}\right)$.

Example. Plot points with these polar coordinates:

$\left(1, \frac{3\pi}{4}\right)$ $\left(-1, \frac{7\pi}{6}\right)$ $\left(3, -\frac{2\pi}{3}\right)$ $\left(-2, -\frac{4\pi}{3}\right)$

Example. Plot regions given by their polar coordinates.

$0 \leq r \leq 1, \quad \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ $-1 \leq r \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{2}$

Usually the polar axis is taken to be the positive x-axis, and the pole is the origin.

In this case, the following formulas give the conversion between polar and Cartesian coordinates.

<u>polar → Cartesian</u>

$x = r\cos\theta$
$y = r\sin\theta$

<u>Cartesian → polar</u>

$r = \sqrt{x^2 + y^2}$
$\tan\theta = \dfrac{y}{x}$
(choose θ so (r, θ) is in correct quadrant)

Example. Convert coordinates.

$\left(2, \dfrac{5\pi}{4}\right) \to$ Cartesian

$(-1, 1) \to$ polar

Polar curves

Example. Draw curves represented by their polar equations.

$r = 3$ $\theta = \dfrac{\pi}{3}$

Typically curves are given by an equation of form $r = f(\theta)$.

We may imagine them as being traced out by a bead sliding on a rotating stick, whose distance from the origin is determined by the function f of θ.

Usually, it is useful to first draw the graph of $r = f(\theta)$ in Cartesian θ-r coordinates to understand how distance from pole increases or decreases as a function of θ.

Example. Sketch the curve $r = 1 - \cos\theta$, called a *cardioid*.

Example. Sketch the curve $r = \sin\theta$. By converting the equation to Cartesian coordinates, verify that the curve is what it looks like.

Example. Sketch the curves $r = \sin(3\theta)$ and $r = \cos(2\theta)$.

In general,

$$\left.\begin{array}{l} r = a\sin(n\theta) \\ r = a\cos(n\theta) \end{array}\right\} \text{ is a rose with } \left\{\begin{array}{l} n \text{ petals, if } n \text{ is odd (traversed once for } 0 \le \theta \le \pi) \\ 2n \text{ petals, if } n \text{ is even (traversed once for } 0 \le \theta \le 2\pi) \end{array}\right.$$

9.4 Areas and Lengths in Polar Coordinates

The area of the region bounded by rays $\theta = a$, $\theta = b$, and the polar curve $r = f(\theta)$, where $a \leq b$ and $f(\theta) \geq 0$ on $[a, b]$, is given by

$$A = \int_a^b \frac{1}{2} f(\theta)^2 \, d\theta.$$

Example. Find the area of the rose with 3 petals, $r = \sin(3\theta)$.

The area of the region bounded by rays $\theta = a$, $\theta = b$ and the polar curves $r = f(\theta)$ and $r = g(\theta)$, where $a \le b$ and $0 \le g(\theta) \le f(\theta)$ on $[a, b]$, is given by

$$A = \int_a^b \frac{1}{2}(f(\theta)^2 - g(\theta)^2)\, d\theta.$$

Example. Find the area inside the curve $r = \sin\theta$ and outside the curve $r = \dfrac{1}{2}$.

Example. Set up the integral for the area inside both curves: $r = 1 - \cos\theta$ and $r = \cos\theta$.

The length of the polar curve $r = f(\theta)$ between $\theta = a$ and $\theta = b$ is given by

$$L = \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_a^b \sqrt{r^2 + (r')^2}\, d\theta.$$

Example. Find the length of the exponential spiral $r = e^\theta$, $0 \le \theta \le 2\pi$.