Calculus 2 — Lecture notes
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8.1 Sequences

Definition. A *sequence* is an infinite list of numbers written in a definite order:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

$$a_n = n\text{-th term}$$

Notation. A sequence is denoted $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples. The following are examples of sequences. Where a formula for the general term of a sequence is not written, write one. Where it is written, write out several terms of the sequence.

a) $1^2, 2^2, 3^2, \dots$

b) $1, \frac{1}{2}, \frac{1}{3}, \dots$

c) $1, -1, 1, -1, \dots$

d) $\left\{\cos \frac{n\pi}{6}\right\}_{n=0}^{\infty} =$

e) $\{n\text{-th digit of } \pi\}_{n=1}^{\infty} =$

f) $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$

g) $1.1, 1.01, 1.001, 1.0001, \dots$

Definition. We say that the sequence $\{a_n\}$ has a limit L and write $\lim_{n\to\infty} a_n = L$ if we can make a_n arbitrarily close to L by taking n sufficiently large. That is, if the numbers a_n get closer and closer to L as n increases.

If $\lim_{n\to\infty} a_n = L$, we say that $\{a_n\}$ *converges* (otherwise, it *diverges*).

$\lim_{n\to\infty} a_n = \infty$ if a_n can be made arbitrarily large by taking n sufficiently large. That is, if the numbers a_n get larger and larger, and without bound, as n increases.

Example. Consider the sequences in the previous example. Which ones converge, and what are their limits?

For a sequence $\{a_n\}$, usually $a_n = f(n)$, where f is some function. For example,

$$a_n = \frac{1}{n} \qquad a_n = \frac{\ln n}{n^2} \qquad a_n = \frac{n^2 - 2n}{2^n}$$

We can use the function $f(x)$ to examine how a_n behaves.

Example. Let $a_n = \dfrac{n^2 + 2n + 3}{n^2 + 8n}$. Examine the function $f(x) = \dfrac{x^2 + 2x + 3}{x^2 + 8x}$ to see if $\lim_{n\to\infty} a_n$ exists.

Theorem. If $\lim_{x\to\infty} f(x)$ exists and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

Example. Find the limit.

$$\lim_{n\to\infty} \frac{\ln n}{n^2} =$$

Example. Find the limit.

$$\lim_{n\to\infty} \cos((2n+1)\pi) =$$

Limits of sequences follow the same limit laws as limits of functions:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ as long as } \lim_{n\to\infty} b_n \neq 0$$

Squeeze Theorem. If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example. $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

Theorem. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example. $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Example.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 \quad \text{if } |r| < 1 \\ 1 \quad \text{if } r = 1 \\ \infty \quad \text{if } r > 1 \\ \text{does not exist, if } r \le -1 \end{cases}$$

Definition.
A sequence is called *increasing* if $a_n \leq a_{n+1}$ for all n, that is $a_1 \leq a_2 \leq a_3 \leq \dots$
A sequence is called *decreasing* if $a_n \geq a_{n+1}$ for all n, that is $a_1 \geq a_2 \geq a_3 \geq \dots$
A sequence is called *monotonic* if it is either increasing or decreasing.

Example. $\{n^2\}$ is ____________ and $\left\{\frac{1}{n}\right\}$ is ____________.

Example. Show that the sequence $\left\{\dfrac{n^2}{n^3+1}\right\}$ is decreasing for $n \geq 2$.

Definition.
A sequence is called *bounded above* if there exists a number M such that $a_n \leq M$ for all n.
A sequence is called *bounded below* if there exists a number m such that $m \leq a_n$ for all n.
A sequence is called *bounded* if it is bounded above and below.

Example. Discuss boundedness of the following sequences:

$\left\{\frac{n^2}{n^3+1}\right\}$ $\qquad$ $\{(-1)^n\}$ $\qquad$ $\{n^2\}$

Theorem. Every bounded monotonic sequence is convergent.

Example. Show that the sequence below is monotonic and bounded, hence has a limit.

$$a_n = \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdot \cdots \cdot \frac{4n^2-1}{4n^2}$$

It turns out that $\lim_{n\to\infty} a_n = \frac{2}{\pi}$.

8.2 Series

We wish to make sense of an infinite sum of numbers:

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

Examples. The following are examples of infinite sums.

a) $1 + 1 + 1 + \cdots = \displaystyle\sum_{n=1}^{\infty} 1$

b) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

c) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

d) $1 - 1 + 1 - 1 + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n$

e) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

f) $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

To understand what infinite sums should mean, recall an improper integral:

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} \int_a^t \frac{1}{x^2}\, dx$$

In a similar spirit, we may define:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

$$a_1 + a_2 + a_3 + \cdots = \lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n)$$

Definition. An infinite sum $a_1 + a_2 + a_3 + \cdots + a_n + \ldots$ is called a *series* and denoted $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$. The number a_n is called the *n-th term of the series.*

We form the sequence of partial sums s_n by adding the first n terms:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n \end{aligned}$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, we say the series $\sum a_n$ is convergent, write $\sum_{n=1}^{\infty} a_n = s$ and call s the *sum of the series.* If $\{s_n\}$ is divergent, we say the series $\sum a_n$ is divergent. (Verbs: converges, diverges.)

Examples. Are the series from the previous examples convergent?

a) $1 + 1 + 1 + \cdots = \sum_{n=1}^{\infty} 1$

b) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ Check numerical evidence.

c) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Check numerical evidence.

d) $1 - 1 + 1 - 1 + \cdots = \sum_{n=0}^{\infty} (-1)^n$

e) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

f) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n}$

This is an example of a *geometric series*, one of form

$$1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n, \text{ or, more generally, } a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

To deal with a geometric series, we first need a fact:

Proposition. $1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$

Theorem. The geometric series $1 + r + r^2 + r^3 + \cdots = \sum_{n=0}^{\infty} r^n$ converges when $|r| < 1$ and its sum is $\frac{1}{1-r}$, and diverges when $|r| \geq 1$. More generally, when $|r| < 1$

$$\sum_{n=k}^{\infty} a\, r^{\text{exponents increasing by 1}} = \frac{\text{first term}}{1-r}$$

Examples. Find the sums.

$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots =$

$\frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \cdots =$

$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}} =$

Example. What numbers do these infinite decimal numbers represent?

$0.22222\cdots =$

$0.99999\cdots =$

Theorem. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Test for Divergence. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Note: theorem does NOT say "if $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges".

For example, $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example. Illustrating convergence of series $\sum a_n$ by considering whether a walker whose n-th step size is a_n approaches a certain point.

Theorem. If $\sum a_n$ and $\sum b_n$ are convergent, so are the series $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ and $\sum ca_n$ and their sums are

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \qquad\qquad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

This follows from limit laws for sequences, because sums of series are limits of sequences of partial sums.

Example. If $\sum a_n$ converges and $\sum b_n$ diverges, what can we say about convergence of $\sum(a_n + b_n)$?

8.3 The Integral and Comparison Tests

In this section we consider series with positive terms $\sum a_n$ (so $a_n > 0$ for every n). Note that in this case $\sum a_n$ either converges, or $\sum a_n = \infty$.

Example. The sum $\sum_{n=1}^{\infty} \frac{1}{n}$ can be interpreted as sum of areas of certain rectangles and compared to area under the function $f(x) = \frac{1}{x}$.

Example. The sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be interpreted as sum of areas of certain rectangles and compared to area under the function $f(x) = \frac{1}{x^2}$.

These examples illustrate a general fact connecting $\sum f(n)$ and $\int_1^{\infty} f(x)\,dx$.

Theorem (The Integral Test). Let f be a continuous, decreasing and positive function on $[1,\infty)$ and let $a_n = f(n)$. Then

1) If $\displaystyle\int_1^\infty f(x)\,dx$ converges, then $\sum a_n$ converges.

2) If $\displaystyle\int_1^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

Example.

Because $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx \begin{cases}\text{converges, if } p>1\\ \text{diverges, if } p\le 1\end{cases}$, the p-series $\displaystyle\sum \frac{1}{n^p} \begin{cases}\text{converges, if } p>1\\ \text{diverges, if } p\le 1\end{cases}$

Example. Determine whether the series converge: a) $\displaystyle\sum \frac{1}{n\sqrt{n}}$ b) $\displaystyle\sum \frac{1}{\sqrt[3]{n}}$

The integral test compared the sum of a series to an area under a function. We can also compare a series to another one, whose convergence is known, to help us determine the convergence of the first series.

Example. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n}$.

Example. Consider $\sum_{n=1}^{\infty} \frac{1}{3^n + 7}$.

Reasoning as in the above two examples, one can prove:

Theorem (The Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with positive terms. Then
1) If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges.
2) If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.

Example. How to handle something like $\sum \frac{n-3}{2n^2 - n}$? Try comparison:

That did not work. Try another idea: isolate dominant terms from the general term of the series.

Theorem (The Limit Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with positive terms, and suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where $c > 0$ is a finite number. Then either both series converge or both diverge. That is, $\sum a_n$ converges if and only if $\sum b_n$ converges.

Example. Determine if $\sum_{n=1}^{\infty} \frac{3n^3 + 4n - 1}{n^6 - n^2 + 1}$ converges.

Example. Determine if $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + n^2}}{3n^{\frac{5}{2}} + n}$ converges.

8.4 Other Convergence Tests

Example. Here are some examples of *alternating series*.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \frac{16}{81} + \cdots = \sum_{n=1}^{\infty} - \left(-\frac{2}{3}\right)^n$$

Definition. A series of type $b_1 - b_2 + b_3 - b_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n > 0$ for all n, is called an *alternating series*.

Theorem (The Alternating Series Test). If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

1) $\{b_n\}$ is decreasing　　　2) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Proof.

Alternating series estimate: $|R_n| = |s - s_n| \le b_{n+1}$.

Example. Show $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges and find the accuracy of s_{10000} in estimating the sum of the series.

Example. Show $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$ converges. How many terms are needed so that s_n approximates the sum with accuracy 10^{-4}?

Given any series $\sum a_n$, we can consider the corresponding series of absolute values:

$$|a_1| + |a_2| + |a_3| + \cdots = \sum_{n=1}^{\infty} |a_n|$$

Definition. A series $\sum a_n$ is *absolutely convergent* if the series of absolute values $\sum |a_n|$ converges.

Example. Determine whether the series converge absolutely:

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}$

b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

Definition. A series $\sum a_n$ is *conditionally convergent* if it is convergent, but not absolutely convergent.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof.

Example. Show the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(n^2+n)}{n^2}$ is convergent.

Theorem (The Ratio Test). Let $\sum a_n$ be a series.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ is absolutely convergent.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_n$ is divergent.

Example. Check the following series for convergence using the ratio test.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

Theorem (The Root Test). Let $\sum a_n$ be a series.
If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum a_n$ is absolutely convergent.
If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ then $\sum a_n$ is divergent.

Example. Check the following series for convergence using the ratio test.

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{4^n}$$

$$\sum_{n=1}^{\infty} \frac{10^n}{n^2 + 17n}$$

Useful limits for the root test:

$\lim_{n\to\infty} \sqrt[n]{a} = 1$ $\qquad$ $\lim_{n\to\infty} \sqrt[n]{n} = 1$ $\qquad$ $\lim_{n\to\infty} \sqrt[n]{P(n)} = 1$, where $P(x)$ is a polynomial

Like the ratio test, the proof of the root test is essentially a comparison to the geometric series.

8.5 Power Series

Definition. A *power series* is a series of form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \dots \text{(centered at 0)}$$

or, more generally,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \dots \text{(centered at } a\text{)}$$

Whether the series converges or not depends on the x we choose; for the x's for which the series converges we get a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Note: the partial sum of a power series is a polynomial.

Example. Which function is given by the series $\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$?

Example. For which x does $\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$ converge?

Example. Find the interval of convergence for $\sum_{n=2}^{\infty} (\ln n)^n x^n$.

Example. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Theorem. For the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are three possibilities:

(i) the series converges only when $x = a$.

(ii) the series converges for all x.

(iii) there is a number R ("radius of convergence") such that

the series converges if $|x-a| < R$

the series diverges if $|x-a| > R$.

(When $|x-a| = R$, that is, when $x = a \pm R$, we have to test separately.)

Example. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(x+1)^n}{4^{n+3}}$.

8.6 Representing Functions as Power Series

Example. Use the geometric series sum

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

to find power series expansions of other functions.

Example. Find the power series expansion for $f(x) = \dfrac{1}{3-x}$ and $g(x) = \dfrac{x^4}{3-x}$. State the interval of convergence.

Example. Find the power series expansion for $f(x) = \dfrac{1}{1+x^2}$. State the interval of convergence.

Series can be integrated or differentiated term-by-term.

Theorem. Suppose the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$. Then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is continuous and differentiable on the interval $(a-R, a+R)$ and:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence for series for f' and $\int f$ are also R.

Example. Differentiate and integrate the power series expansion to get power series expansions for other functions.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$$

Example. Integrate the power series expansion to get something interesting.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n$$

8.7 Taylor and MacLaurin Series

Suppose $f(x)$ has a power series expansion valid on for x satisfying $|x-a|<R$.

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+\dots$$

Calculate $f'(a)$, $f''(a)$, $f'''(a), \dots$

Theorem. If f has a power series representation for x satisfying $|x - a| < R$, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $c_n = \frac{f^{(n)}(a)}{n!}$. Thus, if f has a power series expansion around a, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example. Assume e^x has a power series expansion at 0. Find its power series expansion.

Example. Assume $\sin x$ has a power series expansion at 0. Find its power series expansion.

Example. Differentiate the power series expansion of $\sin x$ to get the power series expansion of $\cos x$.

Note.
Even functions only have even exponents in their power series expansion at 0.
Odd functions only have odd exponents in their power series expansion at 0.

If a function can be differentiated infinitely many times, we can always form a power series just by writing

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called the *Taylor series of f at a.* If $a = 0$, it is called the *MacLaurin series.*

The main question is whether the series converges to the function (usually does), and on which interval.

Definition. The n-th Taylor polynomial of f at a is the n-th partial sum of the Taylor series, namely:

$$T_n(x) = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Definition. Let $R_n(x) = f(x) - T_n(x)$, so that $f(x) = T_n(x) + R_n(x)$. If the function has a power series expansion, then $R_n(x)$ is the n-th remainder of the Taylor series, just like $T_n(x)$ is the n-th partial sum. If $\lim_{n\to\infty} R_n(x) = 0$ then the series converges to $f(x)$.

Example. Draw $T_2(x)$, $T_4(x)$, $T_6(x)$ and $T_8(x)$ for $\cos x$.

Taylor's formula. If f has $n+1$ derivatives in an interval I containing a, then for any x in I there is a number z between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

(Note the remainder is like the first ignored term in the Taylor series except with a z in place of a.)

Note. This sounds like the Mean Value Theorem. Actually, for $n = 0$, this *is* the Mean Value Theorem.

Estimate of $R_n(x)$. If $|f^{(n+1)}(x)| < M$ for all x in I, then for any x in I

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Note. Useful limit : $\lim_{n\to\infty} \frac{r^n}{n!} = 0$, to help show $\lim_{n\to\infty} R_n(x) = 0$.

Example. Show that $\sin x$ is the sum of its Taylor series.

Example. Use series to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Example. Use series to estimate $\sin 20^\circ$ with accuracy 10^{-6}.

Example. Use series to estimate $\int_0^1 \cos x^2\, dx$ with accuracy 10^{-6}.

Example. Find the MacLaurin series for $f(x) = (1+x)^k$ and state its interval of convergence.

The Binomial Series. For any number k and $|x| < 1$,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots,$$

where $\binom{k}{n} = \dfrac{k(k-1)\cdot \dots \cdot (k-(n-1))}{n!}$.

Example. Use known series to find the first five terms of the power series representation of $f(x) = \dfrac{e^x}{1+x}$.

8.8 Applications of Taylor Polynomials

Recall that the Taylor polynomial of f at a is given as

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

It is useful in approximating functions.

Note. For the Taylor polynomial $T_n(x)$ at a,

$$T_n(a) = f(a),\ T_n'(a) = f'(a),\ T_n''(a) = f''(a),\ T_n'''(a) = f'''(a), \ldots, T_n^{(n)}(a) = f^{(n)}(a),$$

so it is not surprising that it approximates the function f well near a.

Example. Let $f(x) = \sqrt{x}$.
a) Find $T_4(x)$ for this function at $a = 9$.
b) Write the expression for the error $R_4(x)$, and estimate it on the intervals $[7, 11]$ and $[8, 10]$.
c) Graph $R_4(x)$ and verify your findings from b).

Example. Let $f(x) = \sin x$.
a) Write $T_5(x)$ for this function at $a = 0$.
b) What is the accuracy or $T_5\left(\frac{1}{2}\right)$?
c) What is the accuracy or $T_5(x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?
d) For which n will $T_n(x)$ approximate $\sin x$ with accuracy 10^{-5} on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?