7.1 Area Between Curves

Example. Find the area of the region between the curves $y = f(x)$ and $y = g(x)$, if $f(x) \geq g(x)$ and $a \leq x \leq b$.

If $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ is given by

$$A = \int_a^b f(x) - g(x)\,dx$$

Note: Always draw a picture first.

Example. Find the area of the region between the curves $y = x^2 - 2x$ and $y = 12 - x^2$.

Example. Find the area of the region between the curves $y = \cos x$ and $y = \sin(2x)$, $x = 0$ and $x = \frac{\pi}{2}$.

In general, area between two curves is $\displaystyle\int_a^b |f(x) - g(x)|\, dx$.

7.2 Volumes

The general principle behind computing volumes:

Consider cross-sections of the solid by planes perpendicular to an axis, and let $A(x)$ be the area of the cross section corresponding to a point x on the axis.

Using areas $A(x)$ we find the approximate volume of the solid:

Then $V \approx \displaystyle\sum_{i=1}^{n} A(x_i^*)\,\Delta x$. Making the subdivisions of $[a, b]$ smaller causes the sum to get closer to the actual volume. However, $\sum_{i=1}^{n} A(x_i^*)\,\Delta x$ is also a Riemann sum for the function $A(x)$ over the interval $[a, b]$, so the Riemann sums approach the integral $\int_a^b A(x)\,dx$. Therefore,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx, \text{ so}$$

$$V = \int_a^b A(x)\,dx, \qquad \text{volume} = \begin{array}{l}\text{integral of}\\ \text{cross-sectional}\\ \text{area}\end{array}$$

Example. Find the volume of the solid obtained by rotating the region between $y = x^3$, $x = 0$, $x = 2$ and $y = 0$ about the x-axis.

Example. Find the volume of the solid obtained by rotating the region bounded by $x = 1$, $x = 4$, $y = x$ and $y = x + 2$ about the x-axis.

Example. Verify that the volume of a ball of radius r is $\dfrac{4}{3}\pi r^3$.

Example. Find the volume of a pyramid with a square base of side a, whose height is h, and whose vertex is directly above one of the vertices of the base.

Note: The same can be shown for a pyramid with any base: $V = \dfrac{1}{3}Bh$.

7.3 Volumes by Cylindrical Shells

Example. Compute the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Using cross-sections perpendicular to axis of rotation y, we run into a problem:

it is hard to solve the equation $y = 2x^2 - x^3$ for x, which is needed to get the radii $r_1(y)$ and $r_2(y)$.

To get around this, use the "shell method," which subdivides the solid into "cylindrical shells." Let $S(x)$ denote the surface area of the shell.

Then $V \approx \sum_{i=1}^{n} S(x_i^*)\,\Delta x$. Making the subdivisions of $[a, b]$ smaller causes the sum to get closer to the actual volume. However, $\sum_{i=1}^{n} S(x_i^*)\,\Delta x$ is also a Riemann sum for the function $S(x)$ over the interval $[a, b]$, so the Riemann sums approach the integral $\int_a^b S(x)\,dx$. Therefore,

$$V = \int_a^b S(x)\,dx, \qquad \text{volume} = \begin{array}{l}\text{integral of}\\ \text{cylindrical shell}\\ \text{surface area}\end{array}$$

Note. In a typical problem, where rotation is done around the y-axis, $S(x) = 2\pi x h(x)$.

Earlier Example. Compute the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.

Remember: always draw the @!♣#◇$ picture!

Example. Find the volume of the solid obtained by rotating the region between $y = 2$, $y = 9 - x$ and $y = 2x$ about the x-axis.

7.4 Arc Length

Problem: Find the length L of the graph of the function $f(x)$ from $x = a$ to $x = b$.

We first approximate the length using line segments:

Then $L \approx \displaystyle\sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\,\Delta x$. Making the subdivisions of $[a, b]$ smaller causes the sum to get closer to the actual length. However, $\sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\,\Delta x$ is also a Riemann sum for the function $\sqrt{1 + f'(x)^2}$ over the interval $[a, b]$, so the Riemann sums approach the integral $\int_a^b \sqrt{1 + f'(x)^2}\,dx$. Therefore,

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

Note: The integral has a square root of something possibly complicated. Prepare to die.

Example. Find the length of the curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$, where $2 \leq x \leq 4$.

Example. Find the circumference of a circle of radius r.

Calculus 2 — Lecture notes
MAT 308, Fall 2021 — D. Ivanšić

7.5 Area of a Surface of Revolution

To approximate the area of a surface of revolution, subdivide the surface into bands.

For every band, Area $\approx l_i \cdot 2\pi r_i = \sqrt{1+f'(x_i^*)^2}\,\Delta x \cdot 2\pi f(x_i^*)$, giving us the approximation and Riemann sum $S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+f'(x_i^*)^2}\,\Delta x$. By usual considerations, surface area is then an integral:

$$S = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

Example. Find the surface area of a sphere of radius r.

In general, since the rotated curve can usually be thought of as a function of x or y, one integrates $\int_a^b 2\pi r l\, dz$, where z is x or y. Now, $l\, dz$ stands for $\sqrt{1+f'(z)^2}\, dz$, which is abbreviated as ds (length element). Thus, the general formula is

$$S = \int_a^b 2\pi r\, ds$$

Example. Find the surface area of the surface obtained by rotating the curve $y = \sqrt{x}$, $1 \le x \le 4$ about the x-axis, treating it
a) as a function of x
b) as a function of y

7.6 Work

Definition. When a constant force F acts on an object that moves a distance d along the line in the same direction as the force, we say that the force has done work W computed by

$$W = F \cdot d$$

Note that no work is done if the object does not move.

Units for work:
1) SI system: F is in Newtons, d is in meters, so W is in Newton-meters (Nm), or Joules (J)
2) US customary system: F is pounds, d is in feet, so W is in foot-pounds, ft-lb

Example. Find the work done by
a) lifting a book of 3kg to a desk 70cm high.
b) lifting a 30lb bucket 6 feet off the ground.

What if force is not constant?

Example. A spring is stretched by length x from its unextended position. The force needed to keep the string extended by x from its unextended position is:

$$\text{Hooke's law for spring: } F(x) = kx$$

In general, suppose force varies with position x as object is moved from $x = a$ to $x = b$. We approximate the work done: $W \approx \sum_{i=1}^{n} F(x_i^*)\,\Delta x$, which is also a Riemann sum for F over the interval $[a, b]$. Increasing the number of subdivisions improves the approximation of work, but it also tends towards the integral, so

$$W = \int_a^b F(x)\,dx$$

Example. Suppose the constant k for a spring is $k = 200\text{N/m}$. What work is done by extending the spring 30cm from its unextended length?

Example. How much work is done by pumping water from a $1\text{m} \times 1\text{m} \times 3\text{m}$ rectangular box tank to a height 4 meters above the tank, where the $1\text{m} \times 3\text{m}$ side is parallel to the ground?