Calculus 2 — Lecture notes **MAT 308, Fall 2021 — D. Ivanšić**

6.0 The Substitution Rule

The substitution rule is a sort of reverse to the chain rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Suppose we need to integrate something that has form

$$\int f(g(x))g'(x)\,dx$$

This is almost like the above, except it has $f(g(x))$ rather than $f'(g(x))$. If we had a function F so that $F' = f$, we could write:

$$\int f(g(x))g'(x)\,dx = \int F'(g(x))g'(x)\,dx = [\text{recognize chain rule}] = F(g(x))$$

Therefore, in order to integrate something of form $\int f(g(x))g'(x)\,dx$, all we need to know is the antiderivative of f. This is captured as the substitution rule

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \qquad \text{once done substitute back } g(x) \text{ for } u$$

Example. $\displaystyle\int \frac{2x - \sin x}{x^2 + \cos x}\,dx =$

Example. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx =$

The substitution rule works in definite integrals as well:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{(always change the bounds)}$$

Example. $\displaystyle\int_0^1 \frac{2x-5}{(x^2-5x+5)^3}\,dx =$

Example. $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^5\theta\,\ln(5+\cos\theta)\,d\theta =$

Example. $\displaystyle\int_{\frac{25}{\pi^2}}^{\frac{400}{\pi^2}} \frac{\cos\dfrac{5}{\sqrt{x}}}{x^{\frac{3}{2}}}\,dx =$

6.1 Integration by Parts

Integration by parts is a rule that is a sort of reverse to the product rule.

It is written usually as

$$\int u\,dv = uv - \int v\,du$$

Formula offers a way to deal with the integral of a product:

Example. $\int xe^x\,dx =$

Example. $\int \ln x\,dx =$

Example. $\int x^2 \sin(2x)\, dx =$

Example. $\int e^{3x} \cos x\, dx =$

Example. (A reduction formula) $\int x^7 e^x\,dx = x^7 e^x - 7\int x^6 e^x\,dx$

Integration by parts also works with a definite integral:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example. $\int_0^1 \arctan x\,dx =$

6.2 Trigonometric Integrals

Example. $\int \sin^2 x \cos^3 x \, dx =$

Same trick works for $\int \sin^m x \cos^n x \, dx$ as long as at least one of m, n is odd.

What if both are even? Use a half-angle formula (x is half of $2x$):

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Example. $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx =$

Example. $\int \sin^2 x \cos^2 x \, dx =$

Example. $\int \tan^4 x \sec^4 x \, dx =$

Example. $\int \tan^5 x \sec^3 x \, dx =$

Given $\int \tan^m x \sec^n x \, dx$ we can perform the above tricks as long as

1. n is even, or
2. m is odd and $n > 0$

When n is odd and m is even, we deal with $\int \tan^m x \sec^n x \, dx$ on a case-by-case basis.

Example. $\int \tan^2 x \sec x \, dx =$

Trigonometric Substitution.

Example. Find the area of a disk of radius 3. It has equation $x^2 + y^2 = 9$, first find area of a quarter of the disk using an integral.

Note: θ should be chosen in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ because that makes $\cos\theta > 0$, so $\cos\theta = \sqrt{1 - \sin^2\theta}$.

Trigonometric substitution exploits trigonometric identities to eliminate the root.

Use	to eliminate root in	via substitution
$\cos^2\theta + \sin^2\theta = 1$	$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$1 + \tan^2\theta = \sec^2\theta$	$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\tan^2\theta = \sec^2\theta - 1$	$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ \theta \in \left(0, \frac{\pi}{2}\right)$

Example. $\int \frac{dx}{x^2\sqrt{4+x^2}} =$

Example. Any quadratic expression can be brought into form $a^2 - x^2$, $a^2 + x^2$ or $x^2 - a^2$ by completing the square.

$$\int \frac{dx}{\sqrt{x^2+6x+8}} =$$

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6.3 Integration of Rational Functions

We wish to integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $P(x)$, $Q(x)$ are polynomials and $\deg P < \deg Q$.
(If not, do long division first.)

The method is:

1) Factor $Q(x)$ into irreducible factors of form $(ax+b)^k$ or $(ax^2+bx+c)^l$.
("Irreducible" = cannnot be factored further)

2) **Fact.** $\frac{P(x)}{Q(x)}$ can be written as a sum of terms of form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^j}, \quad \text{called } \textit{partial fractions}$$

where $ax+b$ and ax^2+bx+c are irreducible factors of $Q(x)$.

Example. $\frac{5x^7+3x^6-4x^3+x^2+1}{(x-1)(x+4)^3(x^2+2x+6)(x^2+5)^2} =$

Notice that in the *partial fraction decomposition* the exponents on denominators $ax+b$ and ax^2+bx+c go from 1 to the exponent with which they appear in the factorization of $Q(x)$.

The next step is to find the unknown coefficients in the numerators. Afterwards, each partial fraction can be integrated.

Example. $\int \frac{x+2}{x^2(x+3)}\,dx =$

Example. $\displaystyle\int \frac{3x^2 - 4x + 5}{(x-1)(x^2+1)}\,dx =$

Example. $\displaystyle\int \frac{dx}{x^2 + a^2} =$

Example. $\displaystyle\int \frac{3x - 2}{x^2 - 4x + 8}\, dx =$

Example. $\displaystyle\int \frac{dx}{(x^2 + 9)^2} =$

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6.5 Approximate Integration

There are plenty of situations when $\int_a^b f(x)\,dx$ cannot be found using the Fundamental Theorem of Calculus. For example, the function $f(x)$ may not have an antiderivative among elementary functions. The following antiderivatives, which exist by the Fundamental Theorem of Calculus,

$$\int e^{x^2}\,dx,\ \int \sin(x^2)\,dx,\ \int \frac{e^x}{x}\,dx,\ \int \frac{\sin x}{x}\,dx,\ \int \sqrt{1+x^3}\,dx$$

are not elementary functions (an elementary function is any combinations of polynomials, roots, trigonometric, expential functions and their inverses).

In such situations we resort to approximate integration. We have already seen in Calculus 1 how $\int_a^b f(x)\,dx$ can be approximated by

L_n, R_n, T_n, M_n : left, right endpoint, trapezoid, midpoint estimates

Example. Estimate $\displaystyle\int_1^2 \frac{1}{x}\,dx$ using trapezoid and midpoint estimates, doubling the number of subintervals every time. Check accuracy of estimate by comparing to exact value $\ln 2$.

n	T_n	M_n	$E_T = \ln 2 - T_n$	$E_M = \ln 2 - M_n$
10				
20				
40				
80				

Observations from the table:

1) The bigger the n, the smaller the error

2) Error of M_n is about two times smaller than for T_n

3) Errors decrease by about a factor of 4 when we double n

4) Errors of M_n, T_n are opposite in sign.

It is possible to give error estimates for T_n, M_n.

Theorem. Let K =maximum of $|f''(x)|$ on the interval $[a, b]$. (Or, take any $K \geq |f''(x)|$ for all x in $[a, b]$.) Then the errors E_T and E_M for trapezoid and midpoint rules satisfy:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example. For the integral $\int_1^2 \frac{1}{x}\, dx$, what are the greatest possible errors for T_n and M_n when $n = 40, 80$?

Example. For the integral $\int_1^2 \frac{1}{x}\, dx$, how big must n be so that M_n has an error smaller than 10^{-6}?

To get even better accuracy from approximate integration, we should approximate curves with something *curvy*, for example, parabolas, i.e. quadratic functions. A parabola is determined by three points, so we replace the function with a parabola through three adjacent points in the equal-length subdivision of the interval.

In general, the area under one parabola depends on the values at the three points it passes through:

$$\int_{x_0}^{x_2} Ax^2 + Bx + C\,dx = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

Combined areas under all the parabolas (n even) is

giving us the Simpson rule:

$$S_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

It turns out that

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

Example. Compute the Simpson rule and its error for $\int_1^2 \frac{1}{x}\,dx$ and $n = 10, 20, 40, 80$.

n	S_n	$E_S = \ln 2 - S_n$
10		
20		
40		
80		

Theorem. Let K =maximum of $|f^{(4)}(x)|$ on the interval $[a, b]$. (Or, take any $K \geq |f^{(4)}(x)|$ for all x in $[a, b]$.) Then the error E_S for Simpson's rule satisfies:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example. For the integral $\int_1^2 \frac{1}{x}\,dx$, what are the greatest possible errors for S_n when $n = 40, 80$?

Example. For the integral $\int_1^2 \frac{1}{x}\,dx$, how big must n be so that S_n has an error smaller than 10^{-6}?

6.6 Improper Integrals

Example. Consider the following functions on the indicated intervals.

$f(x) = \dfrac{1}{x}$ on $[1, \infty)$ $\qquad$ $f(x) = \dfrac{1}{x^2}$ on $[1, \infty)$ $\qquad$ $f(x) = e^x$ on $[-\infty, 0)$

Are the areas under the curves over those intervals finite? How to compute them?

Let $A(t)$ be the area under $\dfrac{1}{x^2}$ on the interval $[1, t]$. It is reasonable to say that area under $\dfrac{1}{x^2}$ on the interval $[1, \infty]$ is finite if $\lim\limits_{t\to\infty} A(t)$ is a finite number.

$$\lim_{t\to\infty} A(t) =$$

Similarly, check areas under $\dfrac{1}{x}$ and e^x:

Previous examples motivate this

Definition. (Improper integral over an infinite interval, Type 1.) Suppose $\int_a^t f(x)\,dx$ exists for every $t \geq a$. Then we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

If the limit exists (is a real number), we say the improper integral is *convergent* (verb: *converges*). If the limit does not exist, we say the improper integral is *divergent* (verb: *diverges*).

$\int_{-\infty}^b f(x)\,dx$ is defined in the same way: $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$

$\int_{-\infty}^\infty f(x)\,dx$ is said to be *convergent* if *both* $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent for some a (may use any a). In this case,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Example. Is the area under $\frac{1}{x}$ over the interval $[0, 1]$ finite?

Example. $\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx =$

Example. For which $p > 0$ does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Conclusion: $\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{array}{l} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{array}$

Note. $\int_a^\infty f(x)\,dx$ converges if and only if $\int_b^\infty f(x)\,dx$ converges

Example. Does $\int_0^\infty \frac{\cos^2 x}{1+x^2}\,dx$ converge?

This is a difficult integral, but we may use:

Comparison Theorem. Suppose f, g are continuous functions and that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a) If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then $\displaystyle\int_a^\infty g(x)\,dx$ is convergent.

b) If $\displaystyle\int_a^\infty g(x)\,dx$ is divergent, then $\displaystyle\int_a^\infty f(x)\,dx$ is divergent

Example. Use comparison to determine if $\displaystyle\int_0^\infty \frac{\cos^2 x}{1+x^2}\,dx$ converges.

Example. Use comparison to determine if $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ converges.