College Algebra — Lecture notes
MAT 140, Fall 2021 — D. Ivanšić

3.1 Complex Numbers

There is no real number such that $x^2 + 1 = 0$.

Graphically: we see that the graph of $y = x^2 + 1$ has no x-intercept.
Algebraically: the solution of $x^2 + 1 = 0$ would be $x = \pm\sqrt{-1}$, which is not defined.

To remedy this, we expand the set of real numbers using an imaginary number i:

$$i = \sqrt{-1}, \text{ so } i^2 = -1$$

Looks bogus? We already used the method of expanding sets of numbers we work with:

Equation	Problem and workaround
$2x = 1$	Can't solve using integers, so we expand the integers with rational numbers (numbers of form $\frac{a}{b}$, where a and b are integers)
$x^2 = 2$	Can't solve using rational numbers, (there are no integers a, b so that $\left(\frac{a}{b}\right)^2 = 2$ so we expand the rational numbers with real numbers

If $p > 0$, we can write: $\sqrt{-p} = \sqrt{(-1)p} = \sqrt{-1}\sqrt{p} = i\sqrt{p}$.

Example. $\sqrt{-36} =$ $\qquad$ $\sqrt{-12} =$

Definition. Complex numbers are numbers of form $a + bi$, where a and b are real numbers.

$$a = \text{ real part of } a + bi, \quad b = \text{ imaginary part of } a + bi$$

Computing with complex numbers: work as though working with expressions with "variable" i, taking into account that $i^2 = -1$.

Examples.
$(2 + 7i) + (1 - 3i) =$

$2i(5 - 3i) =$

$(2 + i)(4 - 3i) =$

Definition. The conjugate of a complex number $a + bi$ is the complex number $a - bi$.

Example. The conjugate of $-3 + 4i$ is The conjugate of $2 - 5i$ is

Note: The product of a complex number with its conjugate is a real number.

$(a + bi)(a - bi) =$

This fact is used to help us divide complex numbers.

Example. $\dfrac{1 - 2i}{3 + 4i} =$

Example. Investigate powers of i and use the results to find i^{25}.

3.2 Quadratic Functions and Equations

Definition. A *quadratic function* is a function of form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

Definition. A *quadratic equation* is an equation of form $ax^2 + bx + c = 0$, where $a \neq 0$.
We have already solved some quadratic equations.

Example. Solve the equations.

$3x^2 - 15 = 0$ $\qquad$ $4x^2 + 28 = 0$ $\qquad$ $x^2 + 3x - 10 = 0$

Example. Solve the equation. It does not factor, but we can use the trick "completing the square" which relies on the formula $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

$x^2 - 6x + 7 = 0$

Example. Solve the equation by completing the square.

$x^2 + 10x + 29 = 0$

Every quadratic equation can be solved by completing the square.

$ax^2 + bx + c = 0$

In the end, we get the **quadratic formula**: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example. Solve the equations.

$x^2 + 2x - 9 = 0$

$3z^2 + 5 = z$

Note. The quadratic equation $ax^2 + bx + c = 0$ has:
two real solutions, if $b^2 - 4ac > 0$
one real solution, if $b^2 - 4ac = 0$
two complex solutions, if $b^2 - 4ac < 0$ (the solutions are conjugate)

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Example. A 13ft ladder leans against a wall. How should it be positioned so that the distance from the top of the ladder to the base of the wall is 5 feet more than the distance from the bottom of the ladder to the base of the wall?

Example. Find the dimensions of a rectangle with area 270in^2 and perimeter 66in.

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3.3 Graphing Quadratic Functions

Example. Use transformations of graphs to graph the quadratic function $f(x) = 2(x-3)^2 + 5$.

In general, the graph of any function $f(x) = a(x-h)^2 + k$ can be obtained in this way — it is a parabola with vertex (h, k).

Example. Graph $f(x) = -3(x+4)^2 + 7$

Every quadratic function $f(x) = ax^2 + bx + c$ can be written in form $f(x) = a(x-h)^2 + k$, making its graph a parabola that opens up when $a > 0$, or down when $a < 0$. We can show this by completing the square.

Example. Graph $y = x^2 + 4x - 7$

To get all the necessary details of the graph of the quadratic function, follow these

Guidelines for graphing a general quadratic function $f(x) = ax^2 + bx + c$

1) Find the y-intercept and the x-intercepts (if any).

2) Find the vertex (h, k): $h = -\dfrac{b}{2a}$, $k = f(h)$.

3) The graph is a parabola that $\begin{matrix}\text{opens up if } a > 0 \\ \text{opens down if } a < 0\end{matrix}$, has axis of symmetry the line $x = -\dfrac{b}{2a}$.

Example. Graph $f(x) = 3x^2 - x - 4$.

Example. Graph $f(x) = -x^2 + 2x - 5$.

Note. The possibilities for the graph are:

Quadratic Functions Apps

Example. An object fired upwards from height s_0 feet at velocity v_0 feet per second has height in feet given by the function $s(t) = -16t^2 + v_0 t + s_0$, where t is in seconds. An arrow is shot upwards from height 10 feet with initial velocity 40 feet per second.
a) Write the function $s(t)$ that describes the height of the arrow.
b) Sketch a rough graph of $s(t)$.
c) When does the arrow reach its maximum height and what is it?
d) When does the arrow fall to height 10 ft? To the ground?

Example. A clothing store sells 80 dress shirts a month when their price is \$50. The manager notices that for every dollar that the price is dropped, 4 extra shirts are sold.
a) For several prices, write the number of dress shirts that are sold at those prices. Then write the formula that relates the number x of shirts sold to the price p of the shirt.
b) Write the formula for revenue from shirt sales as a function of price. At what price should the shirts sell to get maximal revenue?
c) If each shirt costs the store \$20, write the formula for profit from shirt sales as a function of price. At what price should the shirts sell to get maximal profit?

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3.4 Rational and Radical Equations

Rational equations are equations that have rational expressions in them.

Example. Solve the rational equation.

$$\frac{5}{x-3}=\frac{7}{2x+1}$$

Example. Solve the rational equation.

$$\frac{2}{x-7}+\frac{3}{x+3}=\frac{x+13}{x^2-4x-21}$$

When dealing with rational equations that have an x in a denominator, always check whether the solution gives a zero in the denominator of the original equation. If it does, it is not a solution to the original equation.

Example. Solve the equation involving a radical.

$x - 2 = \sqrt{5x + 34} - 4$

If in the process of solving an eqation we square it, solutions that are not solutions of the original equation may emerge. Therefore, always check that the solutions you get satisfy the original equation.

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3.5 Equations and Inequalites with Absolute Value

Recall: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$ ($|x|$ forgets the sign of x if it had one.)

Properties of absolute value.

$$|ab| = |a||b| \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \qquad |-a| = |a|$$

Recall: $|a - b|$ = distance from a to b, so $|x|$ = distance from x to 0.

Example. Solve the equation $|x| = 5$.

Example. Solve the equation $|3x - 7| = 2$.

Note. $|x| = a$ is equivalent to $x = a$ or $x = -a$.

Example. Solve the inequality $|x| \leq 4$. Read it as "distance from x to 0 is ≤ 4".

Example. Solve the inequality $|x| > 6$. Read it as "distance from x to 0 is > 6".

Example. Solve the inequality $|x - 4| \geq 7$. Read it as "distance from x to 4 is ≥ 7".

Example. Solve the inequality $|x + 1| < 3$. Read it as "distance from x to -1 is < 3".

Example. Solve the inequality $|5 - 3x| > 4$. Read it as "distance from $3x$ to 5 is > 4".

Note. Our way of solving inequalities is a little different from the book because it tries to go around having to memorize the compound inequalities that inequalities with absolute value are equivalent to. If you still prefer to do it that way, here are the equvialent inequalities. For an $a \geq 0$:

$|u| \leq a$ is equivalent to $-a \leq u \leq a$

$|u| \geq a$ is equivalent to $u \leq -a$ or $u \geq a$