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| **Advanced Calculus 2 — Exam 1** |
| **MAT 526/626, Spring 2020 — D. Ivanšić** |

**Name:** ______________________________

*Show all your work!*

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define when a function $f : A \to \mathbf{R}$ is continuous at a point $c \in A$.

**Theory 2.** (3pts) State the Maximum-Minimum Theorem.

**Theory 3.** (3pts) State the theorem that says what is the image of $[a, b]$ under a continuous function.

## Type A problems (5pts each)

**A1.** Prove by definition that the function $f(x) = 7x + 2$ is continuous at every $c \in \mathbf{R}$.

**A2.** Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $A = \{x \in \mathbf{R} \mid f(x)^2 - f^2(x) = 3\}$ (the first one is $f \cdot f$, the second one is $f \circ f$). If $(x_n)$ is a sequence such that $x_n \in A$ and $(x_n)$ converges to $c$, show that $c \in A$.

**A3.** Let $f, g : \mathbf{R} \to \mathbf{R}$ be functions so that $f$ is continuous at a point $c \in \mathbf{R}$ and $g$ is not continuous at $c$. Show that $f + g$ is not continuous at $c$.

**A4.** Prove that the equation $2^x = \sin x$ has infinitely many solutions. (Draw a picture for inspiration.)

**A5.** Use the sequential criterion to show that $f(x) = x^2$ is not uniformly continuous on its domain $\mathbf{R}$.

## Type B problems (8pts each)

**B1.** Prove by definition that the function $f(x) = x^2 - 5x + 3$ is continuous at every $c \in \mathbf{R}$.

**B2.** Let $f : (0, \infty) \to \mathbf{R}$ be defined by $f(x) = x^2$ for a rational $x$, and $f(x) = 3$ for an irrational $x$. Determine the numbers where the function is continuous and where it is not. Justify in detail. You may use the fact that $x^2$ is a continuous function on all reals.

**B3.** Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a $c \in [0, 1]$ such that $f(c) = c$. (To get started, draw a picture.)

**B4.** Let $f, g : [a, b] \to \mathbf{R}$ be Lipschitz functions. Show that the function $f \cdot g$ is Lipschitz.

**B5.** Let $f : [a, b] \to \mathbf{R}$ be continuous and nonconstant, and suppose $f(a) = f(b)$. Show that there is a function value $V \neq f(a), f(b)$ that is taken on at least twice.

**B6.** For $x > 0$, recall that $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$, where the first equation is the definition and the second was proven (so you may use it). Show that rational exponents are well-defined, that is, if $mq = np$, show that $(x^{\frac{1}{n}})^m = (x^{\frac{1}{q}})^p$. Start with $x^{mq} = x^{np}$, raise both sides to $\frac{1}{q}$, and take it from there. This is not hard, but be careful that you not use what you are trying to prove.

## Type C problems (12pts each)

**C1.** Let $f : \mathbf{Q} \to \mathbf{R}$ be a function that, for every $c \in \mathbf{R}$, is uniformly continuous on an interval around $c \in \mathbf{R}$. That is, for every $c \in \mathbf{R}$, there is an interval $(c - \delta_c, c + \delta_c)$ such that the restriction of $f$ to $(c - \delta_c, c + \delta_c) \cap \mathbf{Q}$ is uniformly continuous.
a) Let $c \in \mathbf{R}$ and let $x_n \to c$, where $x_n \in \mathbf{Q}$ for every $n \in \mathbf{N}$. Show that $f(x_n)$ is a Cauchy sequence, hence converges.
b) For $c \in \mathbf{R} - \mathbf{Q}$, define $f(c) = \lim f(x_n)$, where $x_n \in \mathbf{Q}$ is any sequence that converges to $c$. Show that this definition does not depend on the sequence $x_n$, that is, if $x_n, y_n \in \mathbf{Q}$, $x_n \to c$, $y_n \to c$, then $\lim f(x_n) = \lim f(y_n)$.
This allows us to extend the function $f : \mathbf{Q} \to \mathbf{R}$ to all real numbers.

**C2.** Show that the extension of the function $f : \mathbf{Q} \to \mathbf{R}$, as defined above, is uniformly continuous on an interval around every $c \in \mathbf{R}$, and is thus continuous.

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| **Advanced Calculus 2 — Exam 2** |
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*Show all your work!*

*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define when a function $f : I \to \mathbf{R}$ has a relative maximum at a number $c$.

**Theory 2.** (3pts) Define the $n$-th Taylor polynomial at $x_0$ for a function $f$.

**Theory 3.** (3pts) State the Cauchy Mean Value Theorem.

## Type A problems (5pts each)

**A1.** Show this function is differentiable at 0 and find $f'(0)$: $f(x) = \begin{cases} \sin x, \text{ if } x > 0 \\ e^x - 1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \end{cases}$

**A2.** Find the limit: $\lim\limits_{x \to 0+} \ln(x+1) \ln x$.

**A3.** Derive the quotient rule from the product rule as follows: let $h(x) = \frac{f(x)}{g(x)}$, then $h(x)g(x) = f(x)$. Assume $h$ is differentiable and get $h'$ from the second equation after applying the product rule.

**A4.** Use the Mean Value Theorem to show $|\tan x - \tan y| \geq |x - y|$ for all $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

**A5.** Let $f(x) = \sqrt[3]{1+x}$.
a) Write the Taylor polynomials $P_1$ and $P_2$ for $f$ at $x_0 = 0$.
b) Show that for $x > 0$, $P_2(x) \leq \sqrt[3]{1+x} \leq P_1(x)$.

**A6.** Let $I$ be an open interval, $a \in I$, and let $f : I \to \mathbf{R}$ be differentiable. Show: if $\lim\limits_{x \to a} f'(x)$ exists, it is equal to $f'(a)$.

## Type B problems (8pts each)

**B1.** Let $f : [-1, 1] \to \mathbf{R}$ be the piecewise-linear function, where, for $n = 2, 3, \ldots$

$$f(x) = \begin{cases} \text{line between points } \left(\frac{1}{n}, \frac{1}{n^2}\right) \text{ and } \left(\frac{1}{n-1}, \frac{1}{(n-1)^2}\right) \text{ if } x \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \\ \text{line between points } \left(-\frac{1}{n-1}, \frac{1}{(n-1)^2}\right) \text{ and } \left(-\frac{1}{n}, \frac{1}{n^2}\right) \text{ if } x \in \left[-\frac{1}{n-1}, -\frac{1}{n}\right) \\ 0, \text{ if } x = 0. \end{cases}$$

Show that $f$ is differentiable at 0. (Best to use an $\epsilon$-$\delta$ argument and inequality involving the function.)

**B2.** Let $f : [-1, 4] \to \mathbf{R}$ be continuous on $[-1, 4]$ and differentiable on $(-1, 4)$, and suppose that $-2 \le f'(x) \le 7$ for all $x \in (-1, 4)$. If $f(-1) = 21$, use the Mean Value Theorem to establish the range of values that $f(4)$ can take. Give examples to show that the upper and lower bound for $f(4)$ can be achieved.

**B3.** Let $f(x) = x^2 \sin \dfrac{1}{x}$ and $g(x) = x$. Show that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}$ exists, but $\lim\limits_{x\to 0} \dfrac{f'(x)}{g'(x)}$ does not. Does this contradict L'Hospital's rule?

**B4.** Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f'''$ is continuous and for some $c \in \mathbf{R}$, $f'(c) = f''(c) = 0$ and $f'''(c) > 0$. Use Taylor's theorem to show that $f$ does not have a local extreme at $c$.

**B5.** Use a Taylor polynomial to get a rational number (you do not have to simplify it) that approximates $\cos \frac{2}{3}$ with accuracy $10^{-3}$.

**B6.** Show that the equation $x^3 + 3x^2 + 7x + 2 = 0$ has a solution and find an interval in which Newton's method converges regardless of the starting point.

## Type C problems (12pts each)

**C1.** Let $I$ be an open interval and let $f : I \to \mathbf{R}$ be differentiable and convex. Note we are not assuming that $f''$ exists. Show that $f'$ is an increasing function as follows: let $a, b \in I$, $a < b$ and let $x_t = (1-t)a + tb$. Then $f'(a) = \lim\limits_{x_t \to a+} \dfrac{f(x_t) - f(a)}{x_t - a}$. Turn the limit into a limit by $t$ and apply convexity to get an inequality that will help you show $f'(a) < f'(b)$.

| **Advanced Calculus 2 — Exam 3**<br>**MAT 526/626, Spring 2020 — D. Ivanšić** |
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*Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.*

**Theory 1.** (3pts) Define the indefinite integral based at point $a$.

**Theory 2.** (3pts) State the squeeze theorem (criterion of integrability).

**Theory 3.** (3pts) State the first form of the Fundamental Theorem of Calculus (the one dealing with how to compute the integral of a function using an antiderivative).

## Type A problems (5pts each)

**A1.** Give an example of a function that is not Riemann-integrable. Show, using the definition, that it is not Riemann-integrable.

**A2.** Prove the Mean Value Theorem for Integrals: if $f(x)$ is continuous on $[a, b]$, there exists a $c \in [a, b]$ such that $\int_a^b f = f(c)(b - a)$.

**A3.** If $F(x) = \displaystyle\int_{\sqrt{x}}^{e^x} \sin \frac{1}{t^2}\, dt$, find the expression for $F'(x)$.

**A4.** For a function $f : [a, b] \to \mathbf{R}$, suppose there exist sequences of tagged partitions $\dot{\mathcal{P}}_n$, $\dot{\mathcal{Q}}_n$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $||\dot{\mathcal{Q}}_n|| \to 0$, and $|S(f, \dot{\mathcal{P}}_n) - S(f, \dot{\mathcal{Q}}_n)| \geq \frac{1}{10}$ for all $n \in \mathbf{N}$. Show that $f$ is not Riemann-integrable on $[a, b]$.

**A5.** If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

## Type B problems (8pts each)

**B1.** Let $f : [-2, \infty] \to \mathbf{R}$ be the function at right and let $F : [-2, \infty) \to \mathbf{R}$, $F(x) = \int_{-2}^x f$.
a) Calculate $F(x)$.
b) Draw the graphs of $f$ and $F$.
c) Where is $F$ continuous? Differentiable?

$$f(x) = \begin{cases} 3, & \text{if } x \in [-2, 1] \\ 2 + x^2, & \text{if } x \in (1, 2) \\ x, & \text{if } x \in [2, \infty) \end{cases}$$

**B2.** Let $f : [2, 6] \to \mathbf{R}$ be the function at right.
a) Guess the value of $\int_2^6 f$.
b) Prove by definition of the Riemann integral that $\int_2^6 f$ is the number you guessed.

$$f(x) = \begin{cases} -1, & \text{if } x \in [2, 3) \\ 2, & \text{if } x \in [3, 6] \end{cases}$$

**B3.** Let $f : [0,1] \to \mathbf{R}$ be the function at right. Use the squeeze theorem to show $f$ is Riemann-integrable on $[0,1]$.

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0 \end{cases}$$

**B4.** Let $f : [a,b] \to \mathbf{R}$ be an increasing function. Use the squeeze theorem to show $f$ is Riemann integrable on $[a,b]$. (Hint: subdivide the interval $[a,b]$ into $n$ equal subintervals and define functions $\alpha$ and $\omega$ in an easy way on each subinterval to ensure $\alpha \le f \le \omega$.)

**B5.** Let $f \in \mathcal{R}[0,a]$ and let $g : [0,a] \to \mathbf{R}$, $g(x) = f(a-x)$. Use Cauchy's criterion to show that $g \in \mathcal{R}[0,a]$. (Hint: relate tagged partitions and Riemann sums for $g$ to those for $f$.)

## Type C problems (12pts each)

**C1.** Thomae's function $f : [0,1] \to \mathbf{R}$ is given by

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ (reduced) for some } m, n \in \mathbf{N} \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Use the definition to show that $f$ is Riemann-integrable on $[0,1]$, following these steps:
a) For $i \ge 2$, show there are at most $i-1$ reduced fractions in $[0,1]$ with denominator $i$.
b) Given $n \in \mathbf{N}$, let $D_n$ be the set of reduced fractions in $[0,1]$ with denominators $1, 2, \ldots, n$. Show $D_n$ has at most $2 + 1 + 2 + 3 + \cdots + (n-1) = 2 + \frac{n(n-1)}{2}$ elements.
c) Observe that $2 + \frac{n(n-1)}{2} \le \frac{n^2}{2}$ for $n \ge 4$.
d) Let $\dot{\mathcal{P}}_n$ be a tagged partition with $n^3$ equal subintervals and any tags. Show at most $n^2$ of those subintervals contain an element of $D_n$ — the remaining ones do not have any elements of $D_n$ (note an element of $D_n$ may be an endpoint of a subinterval, putting it in two subintervals).
e) Use d) to argue that $S(f, \dot{\mathcal{P}}_n) < \frac{2}{n}$. Conclude $f$ is Riemann-integrable.