Advanced Calculus 2 — Final Exam **MAT 526/626, Spring 2015 — D. Ivanšić**

Name: ____________________________

Show all your work!

Do all the theory problems. Then do at least five problems, at least two of which are of type B or C. If you do more than five, best five will be counted.

Theory 1. (3pts) State the ϵ-δ definition of continuity at a point c.

Theory 2. (3pts) State Taylor's theorem.

Theory 3. (3pts) State Cauchy's theorem for Riemann integrability of a function.

Type A problems (5pts each)

A1. Let $f, g : \mathbf{R} \to \mathbf{R}$ be continuous functions such that $f(a) > g(a)$ and $f(b) < g(b)$, $a < b$. Show that there exists a $c \in (a, b)$ such that $f(c) = g(c)$.

A2. Let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $f(x) < 0$, for all $x \in [a, b]$. Show there exists a number $d < 0$ such that $f(x) < d$, for all $x \in [a, b]$.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be such that f''' exists and $f'''(x) < 0$, and consider the Taylor polynomial P_2 for f at x_0, whose graph is a parabola. Use Taylor's theorem to show that the graph of f is above this parabola for $x < x_0$, and below it for $x > x_0$.

A4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is such that f'' exists and for some $a < c < b$ we have $f(a) = f(b) = 0$ and $f(c) > 0$. Show that there exists a point $d \in (a, b)$ such that $f''(d) < 0$ (use convexity).

A5. If $F(x) = \displaystyle\int_{\cos x}^{\sqrt[3]{x}} \sqrt[4]{4x^3 + 1}\, dt$, find the expression for $F'(x)$.

A6. If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

Type B problems (8pts each)

B1. Let $f : [a, b] \to \mathbf{R}$ be a Lipschitz function such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that the function $\frac{1}{f(x)}$ is Lipschitz, too.

B2. Let $f : [a, b] \to \mathbf{R}$ be continuous, and suppose f takes on some value V_1 at least twice. Show that there is another function value $V_2 \neq V_1$ that is taken on at least twice.

B3. Let $f : [a, \infty) \to \mathbf{R}$ be differentiable, $f(a) = b$ and suppose $m_1 \leq f'(x) \leq m_2$, for all $x \in (a, \infty)$. Use the Mean Value Theorem to show that the graph of f must lie between the lines with slopes m_1 and m_2, passing through (a, b). Conversely, does every smooth graph passing through (a, b) between the two lines satisfy $m_1 \leq f'(x) \leq m_2$?

B4. Use a Taylor polynomial for $\ln x$ at $x_0 = 1$ to get a rational number that approximates $\ln \frac{1}{2}$ with accuracy 10^{-4}.

B5. Find the limit: $\lim_{x \to 0+} (-\ln x)^{\ln(x+1)}$. (Note: for small $x > 0$, $\ln x < 0$, so we need a minus to ensure that the base is a positive number).

B6. Let $f : [-1, \infty] \to \mathbf{R}$ be the function at right and let $F : [-1, \infty) \to \mathbf{R}$, $F(x) = \int_{-1}^{x} f$.
a) Calculate $F(x)$.
b) Draw the graphs of f and F.
c) Where is F continuous? Differentiable?

$$f(x) = \begin{cases} -x, & \text{if } x \in [-1, 1] \\ -1, & \text{if } x \in (1, 4] \\ x - 6, & \text{if } (4, \infty) \end{cases}$$

B7. Let $f : [0, 1] \to \mathbf{R}$ be the function at right. Use the squeeze theorem to show f is Riemann integrable on $[0, 1]$.

$$f(x) = \begin{cases} \frac{n-1}{n}, & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), n \geq 2 \\ 1, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \end{cases}$$

B8. For the integral $\int_0^4 e^{x^2}\,dx$, how many subintervals are needed so that the Simpson estimate S_n has accuracy 10^{-3}?

Type C problems (12pts each)

C1. Show that the function $f(x) = \ln x$ is uniformly continuous on the interval $[1, \infty)$.

C2. Let $f : (0, \infty) \to \mathbf{R}$ be differentiable on $(0, \infty)$. If the following statements are true, prove them, otherwise, find a counterexample.
a) If $\lim_{x \to \infty} f(x) = b$, then $\lim_{x \to \infty} f'(x) = 0$.
b) If $\lim_{x \to \infty} f'(x) = b$, where $b > 0$, then $\lim_{x \to \infty} f(x) = \infty$.
c) If $\lim_{x \to \infty} f'(x) = 0$, then $\lim_{x \to \infty} f(x)$ exists, and is a real number.

C3. If p is a polynomial of degree at most 3, show the Simpson approximation S_n is exact as follows, without using the error estimate. First note that it is enough to show this for the case of two subintervals (i.e., for S_2).
a) For each of the functions $f(x) = 1, x, x^2, x^3$ show that

$$\int_{a-h}^{a+h} f(x)\,dx = \frac{1}{3}h(f(a-h) + 4f(a) + f(a+h)).$$

b) Conclude that if p is any polynomial of degree at most 3, $x_0 < x_1 < x_2$, $x_1 = \frac{x_0+x_2}{2}$, and $h = x_1 - x_0$, then

$$\int_{x_0}^{x_2} p(x)\,dx = \frac{1}{3}h(p(x_0) + 4p(x_1) + p(x_2)),$$

which proves the Simpson approximation is exact for S_2.
c) Conclude the Simpson approximation S_n is exact for any polynomial p of degree at most 3, and any even n.