Advanced Calculus 2 — Exam 3
MAT 526/626, Spring 2015 — D. Ivanšić

Name: ______________________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Assuming tagged partitions have been defined, define the Riemann integral of a function $f : [a, b] \to \mathbf{R}$.

Theory 2. (3pts) State the second form of the Fundamental Theorem of Calculus (the one dealing with differentiability at a point).

Theory 3. (3pts) Explain how the Simpson rule is computed (formula) and what it represents geometrically.

Type A problems (5pts each)

A1. One integral below can be evaluated using the substition theorem, and the other cannot. Evaluate the one that can, and explain why the the substitution theorem cannot be used on the other.

a) $\displaystyle\int_{\ln(\pi/4)}^{\ln(\pi/2)} e^x \sin e^x \, dx$ b) $\displaystyle\int_0^1 \frac{\ln x}{x} \, dx$

A2. If $F(x) = \displaystyle\int_{\sin x}^{\ln x} \frac{1}{1+t^8} \, dt$, find the expression for $F'(x)$.

A3. Let $f : [0, 1] \to \mathbf{R}$ be the function at right. Use Cauchy's criterion to show f is not Riemann integrable.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Q} \\ -1, & \text{if } x \notin \mathbf{Q} \end{cases}$$

A4. If $f : [0, 1] \to \mathbf{R}$ is continuous and has the property $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

A5. Show: if f is Riemann integrable on $[a, b]$ and $\dot{\mathcal{P}}_n$ is a sequence of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, then $S(f, \dot{\mathcal{P}}_n) \to \int_a^b f$.

A6. Write the specific expression (i.e., with numbers, not variables) for the midpoint estimate M_4 of the integral $\int_1^3 \frac{1}{x} \, dx$, but do not evaluate it. Determine its accuracy.

Type B problems (8pts each)

B1. Let $f : [1, \infty] \to \mathbf{R}$ be the function at right and let $F : [1, \infty) \to \mathbf{R}$, $F(x) = \int_1^x f$.
a) Calculate $F(x)$.
b) Draw the graphs of f and F.
c) Where is F continuous? Differentiable?

$$f(x) = \begin{cases} x, & \text{if } x \in [1, 2] \\ 1, & \text{if } x \in (2, 3] \\ 4 - x, & \text{if } (3, \infty) \end{cases}$$

B2. Let $f : [1, 5] \to \mathbf{R}$ be the function at right.
a) Guess the value of $\int_1^5 f$.
b) Prove by definition of the Riemann integral that $\int_1^5 f$ is the number you guessed.

$$f(x) = \begin{cases} -3, & \text{if } x \in [1, 2] \\ 4, & \text{if } x \in (2, 5] \end{cases}$$

B3. Let $f : [0, 1] \to \mathbf{R}$ be the function at right. Use the squeeze theorem to show f is Riemann integrable on $[0, 1]$.

$$f(x) = \begin{cases} \frac{n-1}{n}, & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n-1}\right), n \geq 2 \\ 1, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \end{cases}$$

B4. Let $f : [a, b] \to \mathbf{R}$ be a function with the property: given any $\epsilon > 0$, there exists a step function $\phi : [a, b] \to \mathbf{R}$ such that ϕ uniformly approximates f with accuracy ϵ on the interval $[a, b]$ (this means that $|f(x) - \phi(x)| < \epsilon$, for all $x \in [a, b]$). Show that f is Riemann integrable.

B5. Let $f \in \mathcal{R}[-a, a]$, and let f be an even function (note that f need not be continous, so you may not use the substitution theorem). Show that $\int_{-a}^0 f = \int_0^a f$, and conclude that $\int_{-a}^a f = 2\int_0^a f$. (Hint: let $\dot{\mathcal{P}}_n$ be a sequence of tagged partitions of $[0, a]$ such that $||\dot{\mathcal{P}}_n|| \to 0$. Construct tagged partitions $\dot{\mathcal{Q}}_n$ of $[-a, 0]$ such that $S(f, \dot{\mathcal{P}}_n) = S(f, \dot{\mathcal{Q}}_n)$, and use them to show integrals are same.)

B6. For the integral $\int_3^5 x^2 \sin x \, dx$, how many subintervals are needed so that the trapezoid estimate T_n has accuracy 10^{-3}?

Type C problems (12pts each)

C1. If p is a polynomial of degree at most 3, show the Simpson approximation S_n is exact as follows, without using the error estimate. First note that it is enough to show this for the case of two subintervals (i.e., for S_2).
a) For each of the functions $f(x) = 1, x, x^2, x^3$ show that

$$\int_{a-h}^{a+h} f(x)\, dx = \frac{1}{3}h(f(a-h) + 4f(a) + f(a+h)).$$

b) Conclude that if p is any polynomial of degree at most 3, $x_0 < x_1 < x_2$, $x_1 = \frac{x_0+x_2}{2}$, and $h = x_1 - x_0$, then

$$\int_{x_0}^{x_2} p(x)\, dx = \frac{1}{3}h(p(x_0) + 4p(x_1) + p(x_2)),$$

which proves the Simpson approximation is exact for S_2.
c) Conclude the Simpson approximation S_n is exact for any polynomial p of degree at most 3, and any even n.