Advanced Calculus 2 — Exam 2 **MAT 526/626, Spring 2015 — D. Ivanšić**	**Name:** ______________________ *Show all your work!*

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define the derivative of a function $f : I \to \mathbf{R}$, where I is an interval.

Theory 2. (3pts) Define a convex function.

Theory 3. (3pts) State the Mean Value Theorem.

Type A problems (5pts each)

A1. Is this function differentiable at 0: $f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Q} \\ 0, & \text{if } x \notin \mathbf{Q}? \end{cases}$

A2. Find the limits: a) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$ b) $\lim\limits_{x\to 0+} \sqrt[n]{x}\ln x$.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be such that f''' exists and $f'''(x) < 0$, and consider the Taylor polynomial P_2 for f at x_0, whose graph is a parabola. Use Taylor's theorem to show that the graph of f is above this parabola for $x < x_0$, and below it for $x > x_0$.

A4. Let $L(x)$ be a function such that $L'(x) = e^{x^2}$ (it exists, but cannot be written using elementary functions). Write expressions for the derivatives of the following functions:
a) $L(x)^2$ b) $L(L(x))$ c) $L(\sqrt{x})$

A5. Draw two pictures: the first illustrates how Newton's method works in a favorable setting, and the second shows how it can fail (that is, the next iteration is much farther from the solution than the current one).

A6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is such that f'' exists and for some $a < c < b$ we have $f(a) = f(b) = 0$ and $f(c) > 0$. Show that there exists a point $d \in (a, b)$ such that $f''(d) < 0$ (use convexity).

Type B problems (8pts each)

B1. Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Show that f is differentiable at every point, find $f'(x)$, and show that $\lim\limits_{x\to 0} f'(x)$ does not exist, hence $f'(x)$ is not continuous.

B2. Let $f : [a, \infty) \to \mathbf{R}$ be differentiable, $f(a) = b$ and suppose $m_1 \le f'(x) \le m_2$, for all $x \in (a, \infty)$. Use the Mean Value Theorem to show that the graph of f must lie between the lines with slopes m_1 and m_2, passing through (a, b). Conversely, does every smooth graph passing through (a, b) between the two lines satisfy $m_1 \le f'(x) \le m_2$?

B3. Let $f(x) = \sqrt[3]{x}$.
a) Write the Taylor polynomial P_3 for f at $x_0 = 8$.
b) Find the interval around 8 for which you can guarantee that P_3 approximates f with accuracy 10^{-2}.

B4. Use a Taylor polynomial to get a rational number that approximates $\sqrt{e}$ with accuracy 10^{-4}.

B5. Show that the equation $x^3 + 2x^2 - 5 = 0$ has a solution and find an interval in which Newton's method converges, regardless of the starting point. Also, find how many iterations Newton's method will require to achieve accuracy 10^{-3}.

B6. Find the limit: $\lim_{x \to 0+} (-\ln x)^{\ln(x+1)}$. (Note: for small $x > 0$, $\ln x < 0$, so we need a minus to ensure that the base is a positive number).

Type C problems (12pts each)

C1. Let I be an open interval and $f : I \to \mathbf{R}$. Suppose there exists a continuous function $g : I \to \mathbf{R}$ such that for every $u, v \in I$, there exists a c between u and v such that

$$\frac{f(v) - f(u)}{v - u} = g(c).$$

Show that f is differentiable on I and that $f'(x) = g(x)$, which implies that f' is continuous.

C2. Let $f : (0, \infty) \to \mathbf{R}$ be differentiable on $(0, \infty)$. If the following statements are true, prove them, otherwise, find a counterexample.
a) If $\lim_{x \to \infty} f(x) = b$, then $\lim_{x \to \infty} f'(x) = 0$.
b) If $\lim_{x \to \infty} f'(x) = b$, where $b > 0$, then $\lim_{x \to \infty} f(x) = \infty$.
c) If $\lim_{x \to \infty} f'(x) = 0$, then $\lim_{x \to \infty} f(x)$ exists, and is a real number.