Advanced Calculus 2 — Exam 1
MAT 526/626, Spring 2015 — D. Ivanšić

Name: ______________________

Show all your work!

Do all the theory problems. Then do five problems, at least two of which are of type B or C (one if you are an undergraduate student). If you do more than five, best five will be counted.

Theory 1. (3pts) Define a uniformly continuous function.

Theory 2. (3pts) State the sequential criterion for continuity (only the affirmative version).

Theory 3. (3pts) State the Bolzano Intermediate Value Theorem.

Type A problems (5pts each)

A1. Let $f : \mathbf{R} \to \mathbf{R}$ be such that $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ such that $f(x) > 0.9 \cdot f(c)$ for all $x \in V_\delta(c)$.

A2. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and $A = \{x \in \mathbf{R} \mid f(x) \in [3, 5]\}$. If (x_n) is a sequence such that $x_n \in A$ and (x_n) converges to c, show that $c \in A$.

A3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Show: $f(x)$ is continuous at c if and only if the function $g(x) = f(x) + x$ is continuous at c. Don't do anything complicated.

A4. Let $f, g : \mathbf{R} \to \mathbf{R}$ be continuous functions such that $f(a) > g(a)$ and $f(b) < g(b)$, $a < b$. Show that there exists a $c \in (a, b)$ such that $f(c) = g(c)$.

A5. Let $f : [a, b] \to \mathbf{R}$ be a continuous function such that $f(x) < 0$, for all $x \in [a, b]$. Show there exists a number $d < 0$ such that $f(x) < d$, for all $x \in [a, b]$.

Type B problems (8pts each)

B1. Let $f(x) = [[|x|]]$, where $[[x]]$ is the greatest integer operation (so, f is greatest integer of absolute value of x). Determine the numbers where the function is continuous and where it is not. Justify in detail (not just the picture).

B2. Give an example of a function f that is discontinuous, but $f(x)^2$ is continuous. Is there an example where f is discontinuous, but $f(x)^3$ is continuous? Why or why not?

B3. Let $f : [a, b] \to \mathbf{R}$ be continuous, and let M be its maximum value. Show that $M = \sup\{f(x) \mid x \in [a, b] \cap \mathbf{Q}\}$.

B4. Show that the function $f(x) = \frac{1}{5x+3}$ is Lipschitz on the interval $[0, \infty)$.

B5. Let $f : [a, b] \to \mathbf{R}$ be a Lipschitz function such that $f(x) \neq 0$ for all $x \in [a, b]$. Show that the function $\frac{1}{f(x)}$ is Lipschitz, too.

B6. Let $f : [a, b] \to \mathbf{R}$ be continuous, and suppose f takes on some value V_1 at least twice. Show that there is another function value $V_2 \neq V_1$ that is taken on at least twice.

Type C problems (12pts each)

C1. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on its domain $[0, \infty)$.

C2. Show that the function $f(x) = \ln x$ is uniformly continuous on the interval $[1, \infty)$.

C3. Give an alternate proof that an increasing function $f : \mathbf{R} \to \mathbf{R}$ may have only countably many discontinuities: if f is strictly increasing, let $D = \{c \mid f \text{ is not continuous at } c\}$ and define the function $g(x)$ as follows:

$$g(c) = \begin{cases} f(c), \text{ if } \lim_{x\to c-} f(x) = \lim_{x\to c+} f(x) \\ \text{any rational number in the interval } \left[\lim_{x\to c-} f(x), \lim_{x\to c+} f(x)\right], \text{ if } \lim_{x\to c-} f(x) < \lim_{x\to c+} f(x). \end{cases}$$

a) Show that g is strictly increasing.
b) Show that $g(D) \subseteq \mathbf{Q}$.
c) Apply injectivity of g to argue that D is countable.
d) If f is not strictly increasing, apply problem A3 to replace it by a strictly increasing function with the same discontinuity set.