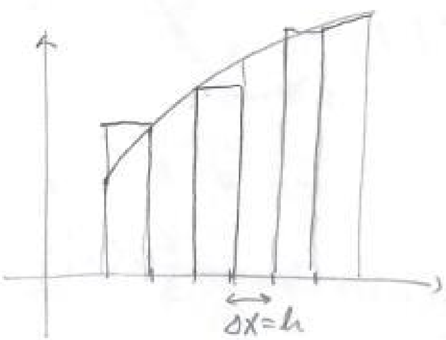
7.5 Approximate Integration

Ex: Evaluate $\int_0^1 \sin x^2\,dx$.

To use FTC, we'd need to find the antider. of $\sin x^2$. However, it's not just hard to find – it's not expressible by elementary functions!

Resort to approximate integration, using certain Riemann sums.

Ex.:



Some possible choices are: $\sum_{i=1}^{n} f(t_i)\cdot h$

L_n: $t_i = x_{i-1}$

R_n: $t_i = x_i$

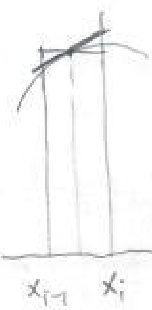
M_n: $t_i = \frac{x_{i-1}+x_i}{2}$

Trapezoid: (Not a Riem. sum)

$$\text{area} = h\cdot\frac{f(x_{i-1})+f(x_i)}{2} = \frac{1}{2}\underbrace{\left(f(x_{i-1})+f(x_i)\right)\cdot h}_{\text{left + right}}$$

$$T_n = \frac{1}{2}(L_n+R_n)$$

Note:



Midpoint is rectangle has same area as "tangent trapezoid" at midpoint.

How good are these estimates?

If f is increasing

$$L_n \le \int_a^b f \le R_n \quad \Big|-T_n$$

$$L_n - \tfrac{1}{2}(L_n+R_n) \le \int_a^b f - T_n \le R_n - \tfrac{1}{2}(L_n+R_n)$$

$$\tfrac{1}{2}(L_n - R_n) \le \int_a^b f - T_n \le \tfrac{1}{2}(R_n - L_n)$$

Since $R_n - L_n = (f(b)-f(a))h$ we get

7.5.1 Theorem If $f:[a,b]\to\mathbb{R}$ is monotone, then

$$\left|\int_a^b f - T_n\right| \le \tfrac{1}{2}\left|f(b)-f(a)\right|\frac{b-a}{n}$$

Ex: $\sin 1^2 - \sin 0^2 \approx 0.85$, $\int_0^1 \sin x^2\,dx$

To get 10^{-3} accuracy for T_n we need

$\frac{1}{2}\,0.85\cdot\frac{1-0}{n} \le 10^{-3}$ $\quad 500\cdot 0.85 \le n$

$425 \le n$

$T_{425} = 0.310268$

7.5.3, 7.5.4, 7.5.6, 7.5.7 Theorem

Let $f:[a,b]\to\mathbb{R}$ be s.t. f'' is cont. on $[a,b]$ and B_2 a number s.t. $|f''(x)| \le B_2$ for all $x\in[a,b]$. Then there exist $c,d\in[a,b]$ s.t.

$$T_n - \int_a^b f = \frac{(b-a)h^2}{12}\cdot f''(c) \qquad M_n - \int_a^b f = \frac{(b-a)h^2}{24}\cdot f''(d)$$

Thus: $\left|T_n - \int_a^b f\right| \le \frac{(b-a)h^2}{12}B_2 = \frac{(b-a)^3}{12n^2}B_2$

$$\left|M_n - \int_a^b f\right| \le \frac{(b-a)h^2}{24}\cdot B_2 = \frac{(b-a)^3}{24n^2}\cdot B_2$$

Pf. for midpoint:

Consider interval $[x_{i-1}, x_i]$

By Taylor's theorem

$$f(x) = \underbrace{f(m_i) + f'(m_i)(x-m_i)}_{l_i(x)} + \frac{f''(c)}{2}(x-m_i)^2$$

If $A \le f''(x) \le B$ on $[a,b]$

we have:

$$\frac{A}{2}(x-m_i)^2 \le f(x) - l_i(x) \le \frac{B}{2}(x-m_i)^2 \quad \int_{x_{i-1}}^{x_i}$$

$$\frac{A}{2}\left(\quad\right) \le \int_{x_{i-1}}^{x_i} f - \text{midpt}_i \le \frac{B}{2}\frac{1}{3}\left((x_i-m_i)^3-(x_{i-1}-m_i)^3\right)$$

$$\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3$$

$$\le \int_{x_{i-1}}^{x_i} - \text{midpt}_i \le B\cdot\frac{1}{6}\,\frac{h^3}{8}\cdot 2$$

$$A\frac{h^3}{24} \le \int_{x_{i-1}}^{x_i} f - \text{midpt}_i \le B\frac{h^3}{24} \quad \Big|\sum_{i=1}^{n}$$

$$A\cdot\frac{h^3}{24}\cdot n \le \int_a^b f - M_n \le B\cdot\frac{h^3}{24}\cdot n \qquad h = \frac{b-a}{n}$$

$$A\frac{(b-a)h^2}{24} \le \int f - M_n \le B\frac{(b-a)h^2}{24}$$

$$A \le \frac{\int f - M_n}{\frac{(b-a)h^2}{24}} \le B \quad \text{there is a } c \in (a,b)$$

s.t. $f''(c) = \dfrac{\int f - M_n}{(\quad)}$ i.e. $\int_a^b f - M_n = f''(c)\dfrac{(b-a)h^2}{24}$

Ex: Error estimate for $n = 425$ and T_{425} gives:

$f' = 2x\cos x^2$ $\qquad |f''| \le 2\cdot(1+2) = 6$

$f'' = 2(\cos x^2 - x\sin x^2 \cdot 2x)$ $\qquad \int_0^1 \sin x^2 - T_{425} \le \dfrac{(1-0)^3}{12\cdot 425^2}\cdot 6$

$= 2(\cos x^2 - 2x^2\sin x^2)$

$$= \frac{1}{2\cdot 425^2} = \frac{1}{361250}$$

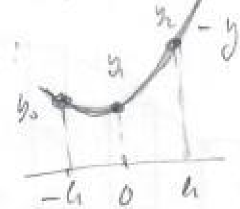
$$= 2.76 \times 10^{-6} < 10^{-5}$$

Simpson's rule:

Subdivide into n subint, n even.

replace function by a parabola on every 'double subinterval'

To get formula, consider a parabola on $[-h, h]$: $\quad y = Ax^2 + Bx + C$



$$\int_{-h}^{h}(Ax^2+Bx+C)\,dx = \left(A\frac{x^3}{3} + Cx\right)\Big|_{-h}^{h}$$

$$= A\cdot\frac{2h^3}{3} + 2Ch = \frac{1}{3}h\left(2Ah^2 + 6C\right)$$

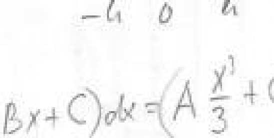
$$= \frac{1}{3}h\left(y_0 + y_2 + 4y_1\right)$$

$C = y_1 \qquad Ah^2 + Bh + C = y_2$

$\qquad\quad Ah^2 - Bh + C = y_0$

$2(Ah^2 + C) = y_0 + y_2$

Now, for any parabola passing through three pts:



$$\int_a^b g(x)\,dx = \begin{bmatrix} x = \frac{a+b}{2} + u & x = b,\ u = \frac{a-b}{2} \\ dx = du & x = a \quad u = \frac{b-a}{2} \end{bmatrix}$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \underbrace{g\left(\frac{a+b}{2} + u\right)}_{\text{quadratic}} dx = \frac{h}{3}\left(g\left(\frac{a+b}{2} - \frac{b-a}{2}\right) + 4g\left(\frac{a+b}{2} + 0\right) + g\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

This gives the pattern for simpson's formula

$$S_n = \frac{1}{3}h\big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\big)$$

<u>7.5.8. and 7.5.9 Theorem</u> Let $f^{(4)}$ be cont. on $[a,b]$, B_4 a number s.t. $|f^{(4)}(x)| \le B_4$. Then there exists a $c \in [a,b]$ s.t.

$$S_n - \int_a^b f = \frac{(b-a)h^4}{180} \cdot f^{(4)}(c)$$

Thus: $\left|S_n - \int_a^b f\right| \le \frac{(b-a)h^4}{180} \cdot B_4 = \frac{(b-a)^5}{180n^4} B_4$

<u>Ex:</u> $f(x) = \sin x^2$

$f'(x) = 2(\cos x^2 - 2x^2 \sin x^2)$

$f''(x) = 2(-2x\sin x^2 - 2(2x\sin x^2 + x^2 \cdot 2x\cos x^2))$

$= 2(-6x\sin x^2 - 4x^3\cos x^2) = -4(3x\sin x^2 + 2x^3\cos x^2)$

$f^{(4)}(x) = -4(3(\sin x^2 + x \cdot 2x\cos x^2) + 2(3x^2\cos x^2 + x^3 \cdot (-2x\sin x^2))$

$= -4(3\sin x^2 + 12x^2\cos x^2 - 4x^4\sin x^2)$

$|f^{(4)}(x)| \le 4(3+12+4) = 56$

If we want accuracy 10^{-5} we need

$\frac{1-0}{180n^4} \cdot 56 < 10^{-5}$ $\quad n = 14$

$\frac{56}{180} \cdot 10^5 < n^4$ $\quad S_{14} = 0.310266$

$31111 < n^4$

$13.28 < n$