ADV. CALCULUS

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## 7.3 The Fundamental theorem

### 7.3.1 The Fund. Theorem of Calculus (First form)

Suppose $E \subseteq [a,b]$ is a finite set and $f, F: [a,b] \to \mathbb{R}$

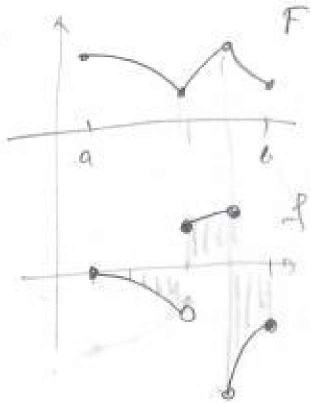
s.t. a) $F$ is cont. on $[a,b]$

b) $F'(x) = f(x)$ for all $x \in [a,b] \setminus E$

c) $f \in \mathcal{R}[a,b]$.

Then $\int_a^b f = F(b) - F(a)$

Ex:



$\int_a^b f = F(b) - F(a) < 0$

Pf. Let $E = \{a, b\}$, in general, break $[a,b]$ into a finite number of intervals.

Given $\varepsilon > 0$, let $\delta$ be s.t. for any TP $\dot{P}$ with $\|\dot{P}\| < \delta$, we have $|S(f,\dot{P}) - \int_a^b f| < \varepsilon$

Applying MVT to $[x_{i-1}, x_i]$ and $F$ we have

$F(x_i) - F(x_{i-1}) = F'(u_i)(x_i - x_{i-1})$ for some $u_i \in I_i$

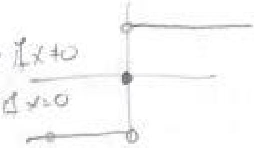
Then $F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = \sum_{i=1}^{n} F'(u_i)(x_i - x_{i-1})$

$= S(f, \dot{P})$ for some tags

Thus $|F(b) - F(a) - \int_a^b f| < \varepsilon$ for every $\varepsilon$

SEWA, $| \quad | = 0$ so $F(b) - F(a) = \int_a^b f$,

Ex: $f(x) = \begin{cases} \operatorname{sign} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



$f \in \mathcal{R}[-3,5]$ and $f(x) = \frac{d}{dx}|x|$ for $x \in [-3,5] \setminus \{0\}$

Thus, FTC applies, and

$\int_{-3}^{5} f = |5| - |-3| = 2$

Ex: $F(x) = \begin{cases} x^2 \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$F'(x) = 2x \cos \frac{1}{x^2} + x^2(-\sin \frac{1}{x^2})(-2x^{-3})$

$= 2x \cos \frac{1}{x^2} + \frac{1}{x} \sin \frac{1}{x^2}$

$F'$ is not bounded on $[0,1]$, so

$\int_0^1 F'$ cannot be found using the FTC.

Def. 7.3.? If $f \in \mathcal{R}[a,b]$ we can define

$F(x) = \int_a^x f$ for $x \in [a,b]$.

$F$ is called the indefinite integral of $f$ with base point $a$.

7.3.4 Theorem The indefinite integral $F$ is continuous on $[a,b]$. Furthermore, if $|f(x)| \leq M$ on $[a,b]$, then $|F(x)-F(u)| \leq M|x-u|$ for all $x,u \in [a,b]$ (i.e. $F$ is Lipschitz.)

Pf. For $u \leq x$ we have

$$F(x) = \int_a^x f = \int_a^u f + \int_u^x f = F(u) + \int_u^x f$$

$$F(x) - F(u) = \int_u^x f$$

so if $|f| \leq M$ we have

$$-M(x-u) \leq \int_u^x f \leq M(x-u)$$

i.e. $|F(x)-F(u)| = |\int_u^x f| \leq M|x-u|$

7.3.5 Fundamental Theorem of Calculus (Second Form)

If $f \in R[a,b]$ cont. at $c$, then the indefinite integral $F$ is differentiable at $c$ and $F'(c) = f(c)$

Pf. Let $c \in [a,b)$, consider the right derivative.
Given $\varepsilon$, then exists a $\delta$ s.t. if $x \in [c, c+\delta)$ then $f(c)-\varepsilon < f(x) < f(c)+\varepsilon$

Let $0<h<\delta$. Then $F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f$

$$h(f(c)-\varepsilon) \leq \int_c^{c+h} f \leq h(f(c)+\varepsilon)$$

so $$f(c) - \varepsilon \leq \frac{F(c+h)-F(c)}{h} \leq f(c)+\varepsilon$$

i.e. $$\left|\frac{F(c+h)-F(c)}{h} - f(c)\right| \leq \varepsilon$$

Since $\varepsilon$ was arbitrary, we showed

$$\lim_{h\to 0+} \frac{F(c+h)-F(c)}{h} = f(c)$$

Similarly for $\lim_{h \to 0-}$.

7.3.6 Theorem If $f$ is cont. on $[a,b]$ then $F$ is diff. on $[a,b]$ and $F' = f$.

Ex: $f(x) = \operatorname{sgn} x$

Considering basepoint $x = -2$

$$F(x) = \int_{-2}^x f = |x| - 2$$

Since $f$ is not cont. at $x=0$, we can't claim that $F'(0) = f(0)$

(Actually $F'(0)$ d.n.e.)

7.3.8 Substitution Theorem Let $J = [\alpha, \beta]$, $\varphi: J \to \mathbb{R}$ has a continuous derivative, $\varphi(J) \subseteq I$ then if $f: I \to \mathbb{R}$ is cont. and

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Pf. Let $F$ be the antiderivative of $f$ on $I$ (for example $F(x) = \int_{\varphi(\alpha)}^x f$). Then $F' = f$

$(F \circ \varphi)' = F'(\varphi(t)) \cdot \varphi'(t)$,

$$\int \underbrace{f(\varphi(t))\varphi'(t)}_{(F\circ\varphi)'(t)}\,dt = (F\circ\varphi)(\beta) - (F\circ\varphi)(\alpha)$$

$$= F(\varphi(\beta)) - F(\varphi(\alpha))$$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} F'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$$

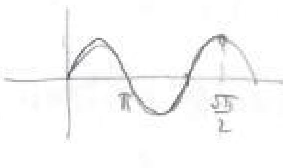
Ex: Note that $\varphi$ need not be injective.

$$\int_0^{\frac{5\pi}{2}} \frac{\cos t}{2+\sin t}\,dt = \int_0^1 \frac{1}{2+x}\,dx = \ln(2+x) = \ln 3 - \ln 2 = \ln\frac{3}{2}$$

$\varphi(t) = \sin t$

$\alpha = 0 \quad \sin 0 = 0$

$\beta = \frac{5\pi}{2} \quad \sin\frac{5\pi}{2} = 1$



$\frac{1}{2+\sin t}(\sin t)'$

Ex: $$\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt = \int_0^4 \frac{2\sin\sqrt{t}}{2\sqrt{t}}\,dt = \begin{bmatrix} x=\sqrt{t} & t=4 \ x=2 \\ dx = 2\sqrt{t}\,dt & t=0 \ x=0 \end{bmatrix}$$

$$= \int_0^2 2\sin x\,dx = -2\cos x\Big]_0^2 = 2(1-\cos 2)$$

could not really use the substitution theorem since $\varphi'(t) = \frac{1}{2\sqrt{t}}$ is not cont. on $[0,1]$.

$(\varphi(t) = \sqrt{t})$

Computation does work for other reasons.

7.3.10

Def: a) A set $Z \subseteq \mathbb{R}$ is a null set (not $\phi$) if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k,b_k)\}_{k=1}^{\infty}$ of open intervals s.t.

$$Z \subseteq \bigcup_{k=1}^{\infty}(a_k,b_k) \text{ and } \sum_{k=1}^{\infty}(b_k-a_k) < \varepsilon$$

b) a statement about $x \in I$ holds almost everywhere if it holds for all $x \in I \setminus Z$, where $Z$ is a null set.

Ex: $\{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ is a null set.

Let $(a_n, b_n) = \left(\frac{1}{n} - \frac{\varepsilon}{2^n}, \frac{1}{n} + \frac{\varepsilon}{2^n}\right)$

$$b_n - a_n = \frac{2\varepsilon}{2^n}$$

$$\sum_{n=1}^{\infty}(b_n-a_n) = \sum_{n=1}^{\infty}\frac{2\varepsilon}{2^n} = 2\varepsilon\sum_{n=1}^{\infty}\frac{1}{2^n} = 2\cdot 1 = 2\varepsilon$$

Ex: $\mathbb{Q}_1 = [0,1] \cap \mathbb{Q}$ is a null set

$\mathbb{Q}_1 = \{r_1, r_2, \dots\}$

$$(a_k, b_k) = \left(r_k - \frac{\varepsilon}{2^k}, r_k - \frac{\varepsilon}{2^k}\right)$$

## 7.3.12 Lebesgue's integrability criterion

A bounded function $f: [a,b] \to \mathbb{R}$ is Riem-integrable iff it is continuous almost everywhere on $[a,b]$.

Ex: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

is not R-int (it is not cont on all of $[0,1]$)

Ex: $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } \mathbb{N} \\ 0 & \text{if otherwise} \end{cases}$

is R-integrable.

## 7.3.14 Composition Theorem

Let $f \in \mathcal{R}[a,b]$, $f([a,b]) \subseteq [c,d]$, $\varphi: [c,d] \to \mathbb{R}$ continuous. Then $\varphi \circ f \in \mathcal{R}[a,b]$.

Pf: (clear if $f$ is cont. on $[a,b]$)

$f \in \mathcal{R}[a,b] \Rightarrow f$ is not cont. on a null set $D$.

$\Rightarrow \varphi \circ f$ is not cont. on a null set $D_1 \subseteq D$.

so $\varphi \circ f \in \mathcal{R}[a,b]$.

7.3.15 Corollary Suppose $f \in \mathcal{R}[a,b]$. Then $|f| \in \mathcal{R}[a,b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a)$$

Pf. Since $|\ |$ is cont., $|f| \in \mathcal{R}[a,b]$ by the composition theorem. Since $-|f| \le f \le |f|$ we get $-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$ so $\left|\int_a^b f\right| \le \int_a^b |f|$. Other inequality follows from $|f| \le M$.

7.3.16 The Product Theorem

If $f, g \in \mathcal{R}[a,b]$, then $f \cdot g \in \mathcal{R}[a,b]$

Pf. $f g = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$

$f^2 \in \mathcal{R}[a,b]$ by Composition Thm.

7.3.17 Integration by Parts $F, G$ diff. on $[a,b]$ and $F' = f$, $G' = g$ are in $\mathcal{R}[a,b]$. Then

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

(This is a partial inverse to the product rule)

We know $(FG)' = F'G + FG' = fG + Fg$

$$\int_a^b (FG)' = \int_a^b fG + \int_a^b Fg$$

$$FG\Big|_a^b = \int_a^b fG + \int_a^b Fg$$

(also true for indefinite integrals)

7.3.18 Taylor's theorem with remainder in integral form

Suppose $f', \dots, f^{(n+1)}$ exist, $f^{(n+1)} \in \mathcal{R}[a,b]$.

Then: $f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n$

where $R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n \, dt$

$$\begin{aligned} f(b) &= f(a) + f(b) - f(a) \\ &= f(a) + \int_a^b f'(t)\,dt = \begin{bmatrix} u = f' & dv = 1\,dt \\ du = f'' & v = -(b-t) \end{bmatrix} \\ &= f(a) + f'(t)\left(-(b-t)\right)\Big|_a^b + \int_a^b f''(t)(b-t)\,dt \\ &= f(a) + f'(a)(b-a) + f''(t)\left(-\frac{(b-t)^2}{2}\right)\Big|_a^b \\ &\quad + \int_a^b f'''(t)\frac{(b-t)^2}{2}\,dt \\ &= f(a) + f'(a)(b-a) + \frac{f''(t)}{2}(b-a)^2 + \\ &\quad + f'''(t)\left(-\frac{(b-t)^3}{3!}\right)\Big|_a^b + \int_a^b f^{(4)}(t)\frac{(b-t)^3}{3!}\,dt \text{ etc.} \end{aligned}$$

Recall error from Taylor's theorem:

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \quad \text{where } c \in (a,b)$$

Now $\frac{1}{n!}\int_a^b f^{(n+1)}(t)\underbrace{(b-t)^n}_{>0}\,dt$

$$= \frac{1}{n!} f^{(n+1)}(c) \int_a^b (b-t)^n\,dt = \frac{1}{(n+1)!} f^{(n+1)}(c)(b-a)^{n+1}$$

there exists a $c \in (a,b)$ Exerc. 7.2.17

(end)