7.2 Riemann Integrable Functions

7.2.1 Cauchy Criterion $f:[a,b]\to\mathbb{R}$ is in $\mathcal{R}[a,b]$ iff for every $\varepsilon>0$ there exists a $\delta>0$ s.t. if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any TP of $[a,b]$ with $\|\dot{\mathcal{P}}\|,\|\dot{\mathcal{Q}}\|<\delta$ then $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|<\varepsilon$.

Pf $\Rightarrow$) Given ε, there exists a $\delta>0$ s.t. if $\dot{\mathcal{P}},\dot{\mathcal{Q}}$ are TP with $\|\dot{\mathcal{P}}\|,\|\dot{\mathcal{Q}}\|<\delta$, then

$$|S(f,\dot{\mathcal{P}})-\int_a^b f|<\frac{\varepsilon}{2}$$
$$|S(f,\dot{\mathcal{Q}}-\int_a^b f|<\frac{\varepsilon}{2}$$

Then $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|<\varepsilon$

$\Leftarrow$) Given $\varepsilon=\frac{1}{n}$, let δ_n be s.t. if $\|\dot{\mathcal{P}}\|,\|\dot{\mathcal{Q}}\|<\delta_n$, then $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|<\frac{1}{n}$

We may assume $\delta_1\geqslant\delta_2\geqslant\ldots$ (otherwise do $\delta_n'=\min\{\delta_1,\ldots,\delta_n\}$). For each n, take a TP $\dot{\mathcal{P}}_n$ s.t. $\|\dot{\mathcal{P}}_n\|<\delta_n$. If $n>m$ then both $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{P}}_m$ have norms $<\delta_m$

so $\quad |S(f,\dot{\mathcal{P}}_n)-S(f,\dot{\mathcal{P}}_m)|<\frac{1}{m}\quad *$

i.e. $S(f,\dot{\mathcal{P}}_n)$ is a Cauchy sequence, so it converges to an $A\in\mathbb{R}$.

Taking $\lim_m$ of $*$ we get

$$|S(f,\dot{\mathcal{P}}_n)-A|\leq\frac{1}{m}$$

We show $A=\int_a^b f$. Given $\varepsilon>0$, let $\delta=\delta_K$, where $K>\frac{2}{\varepsilon}$ (i.e. $\frac{1}{K}<\frac{\varepsilon}{2}$)

For any TP $\dot{\mathcal{Q}}$ where $\|\dot{\mathcal{Q}}\|<\delta$ we have

$$|S(f,\dot{\mathcal{Q}})-A|<|S(f,\dot{\mathcal{Q}})-S(f,\dot{\mathcal{P}}_K)| + |S(f,\dot{\mathcal{P}}_K)-A|$$
$$<\frac{1}{K}+\frac{1}{K}=\frac{2}{K}<\varepsilon$$

Since ε was arbitrary, we are done

Ex: The Cauchy Criterion is often useful to prove a function is not Riemann integrable:

$f\notin\mathcal{R}[a,b]\Leftrightarrow$ there exists an ε_0 s.t. for every δ there exist TP $\dot{\mathcal{P}},\dot{\mathcal{Q}}$ with $\|\dot{\mathcal{P}}\|,\|\dot{\mathcal{Q}}\|<\delta$ and $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|\geq\varepsilon_0$

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}\qquad \text{Let } \varepsilon_0=\frac{1}{2}$$

For every $\delta>0$ we have TP $\dot{\mathcal{P}}$ with all tags in $\mathbb{Q}$, so $S(f,\dot{\mathcal{P}})=1$; $\dot{\mathcal{Q}}$ ______ not in $\mathbb{Q}$ so $S(f,\dot{\mathcal{Q}})=0$

Then $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|=1>\frac{1}{2}$

7.2.3 Squeeze Theorem $f\in\mathcal{R}[a,b]$ iff for any $\varepsilon>0$, there exist functions $\alpha,\omega\in\mathcal{R}[a,b]$ s.t.

$$\alpha(x)\leq f(x)\leq\omega(x)\quad\text{on } [a,b]$$

and $\displaystyle\int_a^b \omega-\alpha<\varepsilon$

Pf. $\Rightarrow$) Take $\alpha = \omega = f$.

$\Leftarrow$) $\alpha, \omega \in \mathcal{R}[a,b]$. Given ε, there exists a δ s.t. for any TP $\dot{P}$ we have

$$\left|S(\alpha,\dot{P}) - \int_a^b \alpha\right| < \varepsilon \qquad \left|S(\omega,\dot{P}) - \int_a^b \omega\right| < \varepsilon$$

so $$\int_a^b \alpha - \varepsilon < S(\alpha,\dot{P}) \qquad S(\omega,\dot{P}) < \int_a^b \omega + \varepsilon$$

Since $\alpha \le f \le \omega$ we have

$$\int\alpha - \varepsilon < S(\alpha,\dot{P}) \le S(f,\dot{P}) \le S(\omega,\dot{P}) < \int\omega + \varepsilon$$

If $\dot{Q}$ is another TP, we have

$$\int\alpha - \varepsilon < S(f,\dot{P}) < \int\omega + \varepsilon$$

$$\int\alpha - \varepsilon < S(f,\dot{Q}) < \int\omega + \varepsilon$$

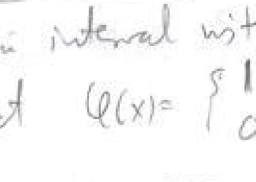
$$\int\alpha - \int\omega - 2\varepsilon < S(f,\dot{P}) - S(f,\dot{Q}) < \int\omega - \int\alpha + 2\varepsilon$$

so $$|S(f,\dot{P}) - S(f,\dot{Q})| < \underbrace{\int\omega - \int\alpha}_{<\varepsilon} + \varepsilon < 3\varepsilon$$

Since ε was a, Cauchy's criterion gives $f \in \mathcal{R}[a,b]$

7.2.4 Lemma Let $J \subseteq [a,b]$ be an interval with endpoints $c \le d$ and set $\varphi(x) = \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \in [a,b] - J \end{cases}$

Then $\varphi \in \mathcal{R}[a,b]$ and $\int_a^b \varphi = d - c$

Pf. 

similar to example we had

7.2.5 Theorem If $\varphi : [a,b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a,b]$

Pf. $\varphi = \sum_{j=1}^{m} k_j \varphi_j$, where φ_j is like in Lemma 7.2.4. Since $\varphi_j \in \mathcal{R}[a,b]$ then so is φ.

7.2.7 Theorem If $f : [a,b] \to \mathbb{R}$ is cont. on $[a,b]$, then $f \in \mathcal{R}[a,b]$.

Pf. f is uniformly cont. on $[a,b]$. Given ε, there exists a δ s.t. $|x-u| < \delta$ then $|f(x) - f(u)| < \frac{\varepsilon}{b-a}$

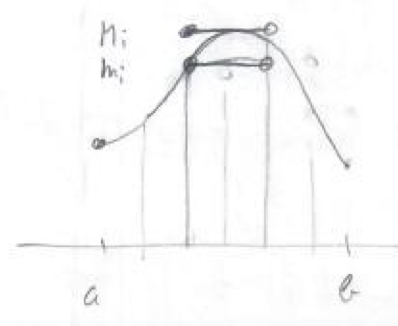
Let $\dot{P}$ be a partition with $\|\dot{P}\| < \delta$ and let $M_i = \max_{x \in I_i} f(x)$, $m_i = \min_{x \in I_i} f(x)$. Note: $M_i - m_i < \frac{\varepsilon}{b-a}$

Form $\alpha(x) = m_i$ if $x \in [x_{i-1}, x_i)$, $\alpha(b) = m_n$

$\omega(x) = M_i$ if $x \in [x_{i-1}, x_i)$, $\omega(b) = M_n$

Clearly $\alpha(x) \le f(x) \le \omega(x)$

and $$\int_a^b (\omega - \alpha) = \sum_{i=1}^{m} (M_i - m_i)\Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \varepsilon.$$



Read 7.2.8 (f monotone $\Rightarrow f \in \mathcal{R}[a,b]$)

<u>7.2.9 Additivity Theorem</u> Let $f:[a,b]\to\mathbb{R}$, $c\in(a,b)$

$f\in\mathcal{R}[a,b] \Leftrightarrow$ restrictions of f to $[a,c]$ and $[c,b]$ are both Riem. integrable

In this case $\int_a^b f = \int_a^c f + \int_c^b f$.

$\Leftarrow$) Let $f_1 = f|_{[a,c]}$, $f_2 = f|_{[c,b]}$ be Riem. int.

Given ε, there exist δ', δ'' s.t. if

if $\dot{P}_1$ is a TP of $[a,c]$ with $\|\dot{P}_1\|<\delta$, then $|S(f_1,\dot{P}_1)-L_1|<\frac{\varepsilon}{3}$

$\dot{P}_2$ ___ $[c,b]$ ___ $\|\dot{P}_2\|<\delta$, then $|S(f_2,\dot{P}_2)-L_2|<\frac{\varepsilon}{3}$

Let M be an upper bound for f on $[a,b]$.

Let $\delta = \min\{\delta', \delta'', \frac{\varepsilon}{6M}\}$

Let $\dot{Q}$ be a TP of $[a,b]$, and $c\in(x_{k-1},x_k)$.

Define partitions

$\dot{Q}_1$ of $[a,c]$: $\{([x_0,x_1],t_1)\ldots([x_{k-2},x_{k-1}],t_{k-1}),([x_{k-1},c],c)\}$

$\dot{Q}_2$ of $[c,b]$: $\{([c,x_k],c),([x_k,x_{k+1}],t_{k+1}),\ldots,[x_{n-1},x_n],t_n)\}$

Then $S(f,\dot{Q})-S(f,\dot{Q}_1)-S(f,\dot{Q}_2)$

$= f(t_k)(x_k-x_{k-1}) - f(c)(c-x_{k-1}) - f(c)(x_k-c)$

$= (f(t_k)-f(c))(x_k-x_{k-1})$

(If $c=x_k$ $S(f,\dot{Q}) = S(f,\dot{Q}_1)+S(f,\dot{Q}_2)$)

$|S(f,\dot{Q})-(L_1+L_2)| \le |S(f,\dot{Q})-S(f,\dot{Q}_1)-S(f,\dot{Q}_2)|$

$+|S(f,\dot{Q}_1-L_1|+|S(f,\dot{Q}_2)-L_2| \le \underbrace{|f(t_k)-f(c)|}_{\le 2M}\underbrace{\Delta x_k}_{<\frac{\varepsilon}{6M}}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$

$< 3\cdot\frac{\varepsilon}{3}=\varepsilon$

$\Rightarrow$) Let $f\in\mathcal{R}[a,b]$. Given $\varepsilon>0$, let δ satisfy the Cauchy criterion.

Let $\dot{P}_1, \dot{Q}_1$ be TP of $[a,c]$ with $\|\dot{P}_1\|, \|\dot{Q}_1\|<\delta$. Extend them to partitions $\dot{P}, \dot{Q}$ of $[a,b]$, $\|\dot{P}\|,\|\dot{Q}\|<\delta$ where tags and subintervals past c are same.

Then $|S(f,\dot{P}_1)-S(f,\dot{Q}_1)|$

$= |S(f,\dot{P})-S(f,\dot{Q})|<\varepsilon$

since $\|\dot{P}\|,\|\dot{Q}\|<\delta$

Hence $f|_{[a,c]}\in\mathcal{R}[a,c]$ by the Cauchy criterion. Similarly for $f|_{[c,b]}$.

Now, first part of proof gives

$$\int_a^b f = \int_a^c f + \int_c^b f$$

<u>7.2.10 Corollary</u> If $f\in\mathcal{R}[a,b]$ and $[c,d]\subseteq[a,b]$ then $f|_{[c,d]}\in\mathcal{R}[c,d]$

<u>Pf:</u> $f|_{[c,d]} = \underbrace{\underbrace{f|_{[c,b]}}_{\text{Riem. int.}}\Big|_{[c,d]}}_{\text{Riem. int.}}$

Def. 7.2.14 If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$

we define: $\int_\beta^\alpha f = -\int_\alpha^\beta f$ and $\int_\alpha^\alpha f = 0$

7.2.13 Theorem If $\alpha, \beta, \gamma \in [a,b]$ and $f \in \mathcal{R}[a,b]$

then $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$

Pf: To avoid considering cases, let

$L(\alpha,\beta,\gamma) = \int_\alpha^\beta + \int_\beta^\gamma + \int_\gamma^\alpha$ Equality above

is equivalent to $L(\alpha,\beta,\gamma) = 0$ for all α, β, γ

It is true when $\alpha < \gamma < \beta$.

Since $L(\alpha,\beta,\gamma) = L(\beta,\gamma,\alpha) = L(\gamma,\alpha,\beta)$

and $L(\beta,\alpha,\gamma) = L(\alpha,\gamma,\beta) = L(\gamma,\beta,\alpha) = -L(\alpha,\beta,\gamma)$

If one of them is 0, they all are,