7.1 Riemann Integral

We wish to precisely define $\int_a^b f(x)dx$.

Def. Let $I=[a,b]$. A <u>partition</u> of I is a finite ordered set $P=(x_0, x_1, \dots, x_n)$, where

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

The points of P divide $[a,b]$ into subintervals $[x_0,x_1], [x_1,x_2] \dots [x_{n-1},x_n]$ of lengths Δx_1 $(=x_1-x_0)$, Δx_2, Δx_n

The <u>norm</u> (<u>mesh</u>) of a partition is
$\|P\| = \max\{\Delta x_1, \dots, \Delta x_n\}$

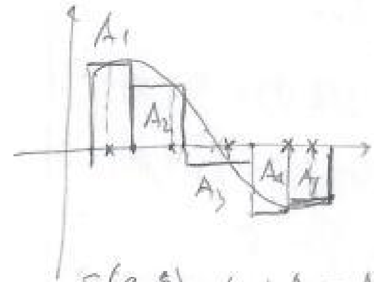
If we choose numbers $t_i \in [x_{i-1}, x_i]$ for $i=1,\dots,n$, then t_i's are called the <u>tags</u> of the subintervals $[x_{i-1},x_i]$.

A tagged partition $\dot{P}$ is a partition P along with a set of tags $t_i \in [x_{i-1}, x_i]$.
(Formally: $\dot{P} = \{(I_i, t_i) \mid i=1,\dots,n\}$)
$\|\dot{P}\|$ is defined as $\|P\|$.

Def. The Riemann sum of a function $f:[a,b]\to\mathbb{R}$ corresponding to the tagged partition $\dot{P}$ is

$$S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

(Use same notation if $\dot{P}$ is a subset of a partition)



$$S(f,\dot{P}) = A_1 + A_2 - A_3 - A_4 - A_5$$

Def. A function $f:[a,b]\to\mathbb{R}$ is <u>Riemann integrable</u> on $[a,b]$ if there exists a number $L\in\mathbb{R}$ s.t. for every $\varepsilon>0$ there exists a $\delta>0$ s.t. if $\dot{P}$ is any tagged partition of $[a,b]$ with $\|\dot{P}\|<\delta$, then $|S(f,\dot{P})-L|<\varepsilon$.

Def. $\mathcal{R}[a,b]$ = set of all Riemann-integrable functions on $[a,b]$.

If f is Riem. integrable, the number L from above is denoted $\int_a^b f$ or $\int_a^b f(x)dx$ and called the <u>Riemann</u> integral of f over $[a,b]$.

<u>Theorem 7.1.2</u> If $f\in\mathcal{R}[a,b]$, then the value of the integral is uniquely determined.

Pf. Let L', L'' satisfy the def, and let $\varepsilon>0$. Then there exists a δ' and δ'' s.t. if $\dot{P}_1, \dot{P}_2$ are any TP s.t. $\|\dot{P}_1\|<\delta'$, $\|\dot{P}_2\|<\delta'$

then $|S(f,\dot{P}_1)-L'|<\frac{\varepsilon}{2}$
$|S(f,\dot{P}_2-L''|<\frac{\varepsilon}{2}$

Let $\delta = \min\{\delta', \delta''\}$, and let $\dot{P}$ be a TP s.t. $\|\dot{P}\| < \delta$. Then

$$|L' - L''| = |L' - S(f,\dot{P}) + S(f,\dot{P}) - L''|$$
$$\le |L' - S(f,\dot{P})| + |S(f,\dot{P}) - L''|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

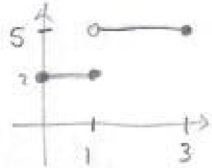
Sεwa we get $|L' - L''| = 0$, so $L' = L''$

__Ex:__ If $f(x) = k$, $f \in \mathcal{R}(a,b)$, $\int_a^b k = k(b-a)$

For any TP $\dot{P}$ we have

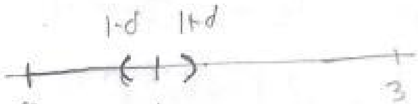
$$S(f,\dot{P}) = \sum_{i=1}^{m} \underbrace{f(t_i)}_{k} \Delta x_i = k \sum_{i=1}^{m} \Delta x_i = k(b-a)$$

__Ex:__ Let $f(x) = \begin{cases} 2, & \text{if } x \in [0,1] \\ 5, & \text{if } x \in (1,3] \end{cases}$



We guess $\int_0^3 f = 12$

Let $\dot{P}$ be a TP, with $\|\dot{P}\| < \delta$, see how this controls $|S(f,\dot{P}) - 12|$



Let $\dot{P}_1$ = subset of $\dot{P}$ whose tags are in $[0,1]$, J_1 = corresp. indices of t_i

$\dot{P}_2$ = $[1,3]$, J_2

Since $\Delta x_i < \delta$ if $t_i \in [0,1]$, then $I_i \subseteq [0, 1+\delta]$

Furthermore, if $u \in [0, 1-\delta]$, then $u \in I_i$ for some i and $I_i \subseteq [0,1]$ so $t_i \in [0,1]$

Thus:

$[0, 1-\delta] \subseteq$ union of subints with tags in $[0,1]$ $\subseteq [0, 1+\delta]$

It follows that

$$S(f,\dot{P}_1) = \sum_{i \in J_1} f(t_i)\Delta x_i = 2 \sum_{i \in J_1} \Delta x_i$$

Since $1-\delta \le \sum \Delta x_i \le 1+\delta$

$$2(1-\delta) \le S(f,\dot{P}_1) \le 2(1+\delta)$$

Similarly,

$[1+\delta, 3] \subseteq$ union of subints with tags in $[1,3]$ $\subseteq [1-\delta, 3]$

$$2-\delta \le \sum_{i \in J_2} \Delta x_i \le 2+\delta$$

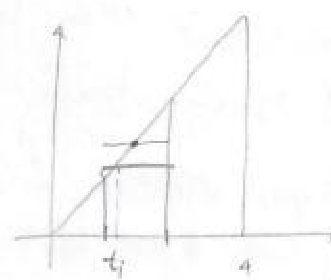
$$5(2-\delta) \le S(f,\dot{P}_2) \le 5(2+\delta)$$

$$12 - 7\delta \le S(f,\dot{P}_1) + S(f,\dot{P}_2) \le 12 + 7\delta$$

$$|S(f,\dot{P}) - 12| \le 7\delta$$

To make this $|\ \ | < \varepsilon$, choose $\delta < \frac{\varepsilon}{7}$.

__Ex:__ $\int_0^4 x\,dx = 8$



Let $\dot{P}$ be a TP with $\|\dot{P}\| < \delta$

Form $\dot{Q} = \{(I_i, q_i = \text{midpt. of } I_i)\}$

$$S(f,\dot{Q}) = \sum_{i=1}^{m} \frac{x_{i-1}+x_i}{2}\Delta x_i = \sum_{i=1}^{m} \frac{x_i^2 - x_{i-1}^2}{2} = \frac{4^2 - 0^2}{2} = 8$$

$$|S(f,\dot{P}) - 8| = |S(f,\dot{P}) - S(f,\dot{Q})| =$$
$$= \left|\sum_{i=1}^{m} t_i \Delta x_i - \sum_{i=1}^{m} q_i \Delta x_i\right| \le \sum_{i=1}^{m} \underbrace{|t_i - q_i|}_{<\delta} \Delta x_i < \delta \cdot 4$$

To make it $< \varepsilon$, choose $\delta = \frac{\varepsilon}{4}$

Ex. Let E = finite subset of k elts of $[a,b]$

$$f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in [a,b] - E \end{cases}$$

Let $\dot{P}$ be a TP with $\|\dot{P}\| < \delta$.

Among the tags, at most $2k$ are equal to an element of E, they are the ones

$$0 \le S(f,\dot{P}) = \underbrace{\sum_{i=1}^{m} f(t_i)\Delta x_i}_{\text{all zero, except at most } 2k} \le 2k \cdot \Delta x_i = 2k\delta$$

$$|S(f,\dot{P}) - 0| \le 2k\delta$$

We can make this $< \varepsilon$ by taking $\delta < \frac{\varepsilon}{2k}$ fixed.

Some properties

7.1.5 Theorem Let $f, g \in \mathcal{R}[a,b]$. Then

a) if $k \in \mathbb{R}$, then $kf \in \mathcal{R}[a,b]$ and $\int_a^b kf = k\int_a^b f$

b) $f+g \in \mathcal{R}[a,b]$ and $\int_a^b f+g = \int_a^b f + \int_a^b g$

c) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$

Pf. If $\dot{P}$ is a TP, then

$$S(kf,\dot{P}) = k\,S(f,\dot{P}), \quad S(f+g,\dot{P}) = S(f,\dot{P}) + S(g,\dot{P})$$

$$S(f,\dot{P}) \le S(g,\dot{P})$$

Given $\varepsilon > 0$, there exists a $\delta >$ s.t. for every $\dot{P}$, $\|\dot{P}\| < \delta$

$$\left|S(f,\dot{P}) - \int_a^b f\right| < \frac{\varepsilon}{2}\left(\frac{\varepsilon}{|k|}\right) \quad \left|S(g,\dot{P}) - \int_a^b g\right| < \frac{\varepsilon}{2}$$

a) Then, for $k \ne 0$ $\quad \left|S(kf,\dot{P}) - k\int_a^b f\right| = \left|k\left(S(f,\dot{P}) - \int_a^b f\right)\right| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$

can be made arb.

ε was arb., done.

b)
$$\left|S(f+g,\dot{P}) - \left(\int_a^b f + \int_a^b g\right)\right|$$
$$\le \left|S(f,\dot{P}) - \int_a^b f\right| + \left|S(g,\dot{P}) - \int_a^b g\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

c) $\int_a^b f - \frac{\varepsilon}{2} < S(f,\dot{P}) \le S(g,\dot{P}) < \int_a^b g + \frac{\varepsilon}{2}$

$\Rightarrow \quad \int_a^b f < \int_a^b g + \varepsilon$

This is true for every ε, so $\int_a^b f \le \int_a^b g$.

7.1.6 Theorem If $f \in \mathcal{R}[a,b]$, then f is bounded on $[a,b]$.

Pf. Given $\varepsilon = 1$, there exists a δ s.t.

$$\left|\sum_{i=1}^{m} f(t_i)\Delta x_i - L\right| < 1 \quad \text{for any TP with } \|\dot{P}\| < \delta$$

$$\Rightarrow \quad \left|f(t_k)\Delta x_k + \sum_{i \ne k} f(t_i)\Delta x_i - L\right| < 1$$

$$|A| - |B| \le |A + B|$$

$$|f(t_k)\Delta x_k| - \left|\sum_{i \ne k} f(t_i)\Delta x_i - L\right| < 1$$

so
$$|f(t_k)| < \frac{1 + \left|\sum_{i \ne k} f(t_i)\Delta x_i - L\right|}{\Delta x_k}$$

for any tagged partition

Take the tagged partition with $\Delta x_i = \frac{b-a}{m}$ where $\frac{b-a}{m} < \delta$

Fixing t_i's, $i \ne k$, we have

$|f(t_k)| \le$ some number for every $t_k \in I_k$

so f is bounded on I_k,

so it is bounded on all of $[a,b]$.