Calculus 2 — Final Exam
MAT 308, Spring 2020 — D. Ivanšić

Name: Saul Ocean

Show all your work!

If you are filming yourself as you take the exam for later upload, write **code: 32FE5Y** on the first sheet of paper with your solutions. Then hold the paper at the beginning so the code can be captured by the camera.

Find the following integrals:

1. (6pts) $\int xe^{2x}\,dx = \begin{bmatrix} u=x & dv=e^{2x}dx \\ du=1dx & v=\frac{e^{2x}}{2}\end{bmatrix}$

$= x\cdot\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx$

$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$

2. (10pts) $\int \sec^4 x\tan^3 x\,dx = \int \sec^2 x\tan^3 x\sec^2 x\,dx$

$= \int(\tan^2 x+1)\tan^3 x\sec^2 x\,dx = \begin{bmatrix} u=\tan x \\ du=\sec^2 x\,dx\end{bmatrix}$

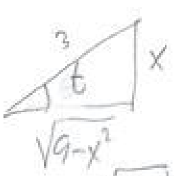
$= \int(u^2+1)u^3\,du = \int u^5+u^3\,du = \frac{u^6}{6}+\frac{u^4}{4} = \frac{\tan^6 x}{6}+\frac{\tan^4 x}{4}+C$

3. (12pts) Use trigonometric substitution to evaluate the integral.

$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \begin{bmatrix} x=3\sin t \\ dx=3\cos t\,dt\end{bmatrix} = \int\frac{(3\sin t)^3}{\sqrt{9-9\sin^2 t}}3\cos t\,dt$

$= \int\frac{3^4\sin^3 t\cos t}{\sqrt{9(1-\sin^2 t)}} = \int\frac{\cancel{3^4}^{\,3}\sin^3 t\,\cancel{\cos t}}{\cancel{3}\,\cancel{\cos t}}\,dt = 27\int\sin^3 t\,dt$

$\cos^2 t$



$\cos t = \frac{\sqrt{9-x^2}}{3}$

$= \int\sin^2 t\sin t\,dt = \int(1-\cos^2 t)\sin t\,dt = \begin{bmatrix} u=\cos t \\ du=-\sin t\,dt \\ -du=\sin t\,dt\end{bmatrix} = \int(1-u^2)(-1)\,du$

u^2-1

$= \frac{u^3}{3}-u = \frac{\cos^3 t}{3}-\cos t = \frac{\left(\frac{\sqrt{9-x^2}}{3}\right)^3}{3} - \frac{\sqrt{9-x^2}}{3} = \frac{(9-x^2)^{\frac{3}{2}}}{81} - \frac{\sqrt{9-x^2}}{3}+C$

4. (6pts) Determine whether the following improper integral converges, and, if so, evaluate it.

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan x \Big|_0^t = \lim_{t\to\infty} \big(\underbrace{\arctan t}_{\to \frac{\pi}{2}} - \underbrace{\arctan 0}_{=0}\big) = \frac{\pi}{2}$$

5. (16pts) The region bounded by the curves $y = x^2 - 4x$ and $y = 30 - x^2$ is rotated around the x-axis.

a) Sketch the solid and a typical cross-sectional washer.

b) Set up the integral for the volume of the solid. Simplify, but do not evaluate the integral.

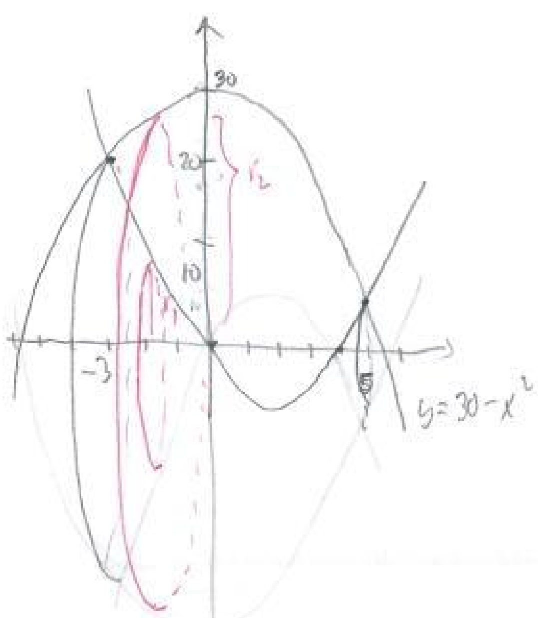
a) $x^2 - 4x = 30 - x^2$

$2x^2 - 4x - 30 = 0 \quad | \div 2$

$x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$x = -3, 5$



b) $\text{Volume} = \int_{-3}^{5} A(x)\,dx = \int_{-3}^{5} \pi r_2^2 - \pi r_1^2\,dx$

$= \int_{-3}^{5} \pi\left((30-x^2)^2 - (x^2-4x)^2\right)dx$

$= \int_{-3}^{5} \pi\left(900 - 60x^2 + x^4 - (x^4 - 8x^3 + 16x^2)\right)dx$

$= \int_{-3}^{5} \pi\left(8x^3 - 76x^2 + 900\right)dx$

Note: This problem has an error, because the region between the curves is not above the x-axis. The above "solution" captures the ideas that would appear in this problem if the region was above x-axis.

6. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{7^{n-1}} = \sum_{n=0}^{\infty} \frac{2\cdot 2^{2n}}{7^n \cdot 7^{-1}} = \sum_{n=0}^{\infty} \frac{2}{7^{-1}} \cdot \frac{(2^2)^n}{7^n} = \left[\text{First term} \cdot \frac{1}{1-r}\right] = \frac{2^1}{7^{-1}} \cdot \frac{1}{1-\frac{4}{7}}$$

converges, because it's a geometric series with $|r|<1$

$r = \frac{2^2}{7} = \frac{4}{7}$

$$= 14 \cdot \frac{1}{\frac{3}{7}} = 14 \cdot \frac{7}{3} = \frac{98}{3}$$

7. (14pts) Find the interval of convergence of the series. Don't forget to check the endpoints.

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^{n+1}(n+4)}.$$

Root test: $\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{(x-3)^n}{2^{n+1}(n+4)}\right|} = \frac{\sqrt[n]{|x-3|^n}}{\sqrt[n]{2^n \cdot 2}\ \sqrt[n]{n+3}}$

$$= \frac{|x-3|}{\underbrace{\sqrt[n]{2^n}}_{=2} \cdot \underbrace{\sqrt[n]{2}}_{\to 1} \cdot \underbrace{\sqrt[n]{n+3}}_{\to 1}} \to \frac{|x-3|}{2}$$

$\frac{|x-3|}{2} < 1 \quad |x-3| < 2$

$|x-3| < 2$

$-2 < x-3 < 2$

$1 < x < 5$

$x = 1$, get $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+1}(n+4)} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}(n+4)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+4)}$

converges by alt. series test

$x = 5$, get $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}(n+4)} = \sum_{n=0}^{\infty} \frac{1}{2(n+4)}$ like $\sum \frac{1}{n}$, diverges by limit comparison test.

Interval of convergence:

$[1, 5)$

8. (16pts) Let $f(x) = \ln x$.

a) Find the 3rd Taylor polynomial for f centered at $a = 10$.

b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $(8, 12)$.

n	$f^{(n)}(x)$	$f^{(n)}(10)$
0	$\ln x$	$\ln 10$
1	x^{-1}	$\frac{1}{10}$
2	$-x^{-2}$	$-\frac{1}{100}$
3	$(-1)(-2)x^{-3}$	$\frac{2!}{1000}$
4	$(-1)(-2)(-1)x^{-4}$	$\frac{-3!}{10000}$

$$T_3(x) = \ln 10 + \frac{\frac{1}{10}}{1!}(x-10) + \frac{-\frac{1}{100}}{2!}(x-10)^2 + \frac{\frac{\cancel{2!}}{1000}}{\cancel{3!}\,3}(x-10)^3$$

$$= \ln 10 + \frac{1}{10}(x-10) - \frac{1}{200}(x-10)^2 + \frac{1}{3000}(x-10)^3$$

$$|R_3(x)| = \left|\frac{f^{(4)}(z)}{4!}(x-10)^4\right| \le \frac{\frac{\cancel{3!}}{8^4}}{\cancel{4!}\,4} \cdot 2^4 = \frac{1}{4\cdot 8^4}\cdot 2^4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^4 = \frac{1}{\underbrace{4^5}_{=2^{10}}} = \frac{1}{1024}$$

$\left|\frac{-3!}{x^4}\right|$ is largest at $x=8$ on $[8,12]$

$|x-10| < 2$ on $[8,12]$

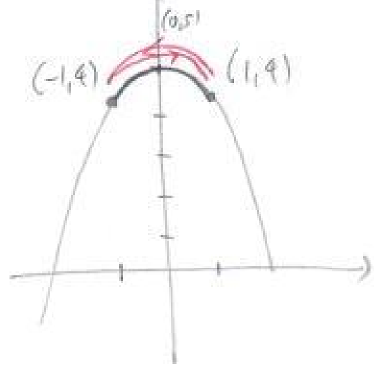
9. (10pts) A particle moves along the path with parametric equations $x(t) = \cos t$, $y(t) = 4 + \sin^2 t$, $0 \le t \le 2\pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = \cos t$

$y = 4 + \sin^2 t = 4 + 1 - \cos^2 t = 5 - x^2$

$y = 5 - x^2$

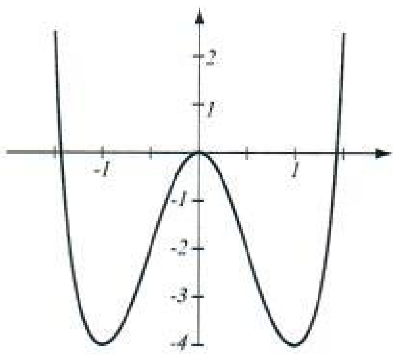
t	x	y
0	1	4
$\frac{\pi}{2}$	0	5
π	-1	4
$\frac{3\pi}{2}$	0	5
2π	1	4



$-1 \le x(t) \le 1$

Moves from $(1,4)$ to $(-1,4)$ and back along the curve $y = 5 - x^2$

10. (24pts) The integral $\int_0^1 \cos(x^2)\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $\cos(x^2)$ is not expressible using elementary functions.
a) Write the expression you would use to calculate M_6, the midpoint rule with 6 subintervals. All the terms need to be explicitly written, do not use f in the sum.
b) The graph of y'' is shown: use it to find the error estimate for M_n in general.
c) What should n be in order for M_n to give you an error less than 10^{-4}?
d) Use a known power series for to find a power series for the above integral.
e) How many terms of the power series are needed to estimate the integral to accuracy 10^{-4}? Write the estimate as a sum (you do not have to simplify it).
f) Which method requires less computation to evaluate the integral with accuracy 10^{-4}, midpoint formula or series?



$K_2 = 4$

a) $\frac{1}{12}\ \frac{3}{12}\ \frac{5}{12}\ \frac{7}{12}\ \frac{9}{12}\ \frac{11}{12}$

$0\ \ \frac{1}{6}\ \ \frac{1}{3}\ \ \frac{1}{2}\ \ \frac{2}{3}\ \ \frac{5}{6}\ \ 1$

$M_6 = \left(\cos\left(\frac{1}{12}\right)^2 + \cos\left(\frac{3}{12}\right)^2 + \cos\left(\frac{5}{12}\right)^2 + \cos\left(\frac{7}{12}\right)^2 + \cos\left(\frac{9}{12}\right)^2 + \cos\left(\frac{11}{12}\right)^2\right)\cdot\frac{1}{6}$

b) error $\le \frac{K_2}{24 \cdot n^2}\cdot(1-0)^3 \le \frac{4}{24n^2} = \frac{1}{6n^2}$ $\qquad \sqrt{6} \approx 2.5$

c) need $\frac{1}{6n^2} \le 10^{-4}$ $\quad n^2 \ge \frac{10^4}{6}$, $n \ge \frac{10^2}{\sqrt{6}} = \frac{100}{\sqrt{6}} \approx 40$

d) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad \int_0^1 \cos x^2\,dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}\Big|_0^1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(2n)!} = \underbrace{1 - \frac{1}{5\cdot 2!} + \frac{1}{9\cdot 4!} - \frac{1}{13\cdot 6!}}_{\text{estimate}} + \frac{1}{17\cdot 8!} - \dots$

e) Need term $< \frac{1}{10^4}$

denom $\ge 10^4$

$5\cdot 2!$	10
$9\cdot 4!$	216
$13\cdot 6!$	9360
$17\cdot 8!$	$> 10^4$

$720\cdot 13$
2160
9360

f) Series requires less computation
(Midpoint would involve 40^+ evaluations of $\cos x$)

11. (10pts) In another attempt to fight the coronavirus, a bottle of disinfectant is thrown from the origin so that its position is given by $x(t) = 15t$, $y(t) = 16t - 5t^2$, where length is measured in meters, time in seconds. Find the equation of the tangent line to this curve when $t = 2$.

$x = 15t \quad y = 16t - 5t^2$

$x' = 15 \quad y' = 16 - 10t$

$t = 2$, $\ x = 30,\ y = 12$
$x' = 15,\ y' = -4$

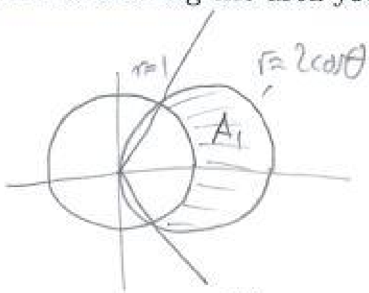
slope $= \frac{y'(1)}{x'(1)} = \frac{-4}{15}$

Eq. of tangent line: $y - 12 = -\frac{4}{15}(x - 30)$

$y = -\frac{4}{15}x + 8 + 12$

$y = -\frac{4}{15}x + 20$

12. (16pts) Find the area inside the polar curve $r = 2\cos\theta$ and outside $r = 1$. Draw a picture showing the area you are computing.



$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

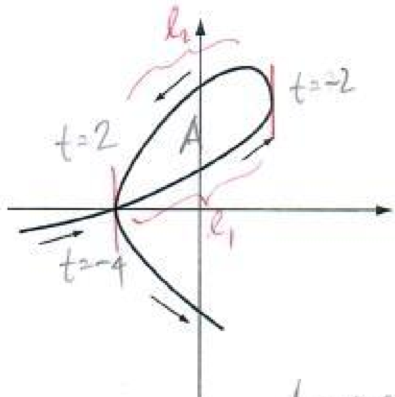
$\theta = \pm\frac{\pi}{6}$

$\frac{1+\cos(2\theta)}{2}$

$A = 2A_1 = 2\int_0^{\pi/6} \frac{1}{2}\left((2\cos\theta)^2 - 1^2\right) d\theta = \int_0^{\pi/6} 4\cos^2\theta - 1 \, d\theta = \int_0^{\pi/6} 2(1+\cos(2\theta)) - 1 \, d\theta$

$= \int_0^{\pi/6} 1 + 2\cos(2\theta) = \frac{\pi}{6} + \sin(2\theta)\Big|_0^{\pi/6} = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$

Bonus. (15pts) The graph of the parametric curve $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$ is shown. Compute the area enclosed by the loop.



$x'(t) = 0$ $\quad$ $y(t) = 0$

$3t^2 - 12 = 0$ $\quad$ $-t^2 - 2t + 8 = 0$

$t^2 = 4$ $\quad$ $t^2 + 2t - 8 = 0$

$t = \pm 2$ $\quad$ $(t+4)(t-2) = 0$

$t = -4, 2$

A = area between l_2 and x-axis − area between l_1 and x-axis

$= -\int_{-2}^{2} y(t)x'(t)dt - \int_{-4}^{-2} y(t)x'(t) = -\left(\int_{-2}^{2} + \int_{-4}^{-2}\right) = -\int_{-4}^{2} (-t^2 - 2t + 8)(3t^2 - 12)dt$

$= \int_{-4}^{2} (t^2 + 2t - 8)3(t^2 - 4)\,dt = 3\int_{-4}^{2} t^4 - 4t^2 + 2t^3 - 8t - 8t^2 + 32\,dt = 3\int_{-4}^{2} t^4 + 2t^3 - 12t^2 - 8t + 32\,dt$

$= 3\left(\frac{t^5}{5} + \frac{t^4}{2} - 4t^3 - 4t^2\Big|_{-4}^{2} + 32\cdot(2-(-4))\right) = 3\left(\frac{32+1024}{5} + \frac{16-256}{2} - 4(8+64) - 4(4-16) + 192\right)$

$= 3\left(\frac{1056}{5} - 120 - 288 + 48 + 192\right) = 3\left(\frac{1056}{5} - 408 + 240\right) = 3\left(211\tfrac{1}{5} - 168\right) = 3\left(43\tfrac{1}{5}\right)$

$= 129\tfrac{3}{5}$