Calculus 2 — Exam 5
MAT 308, Spring 2020 — D. Ivanšić

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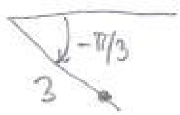
Show all your work!

If you are filming yourself as you take the exam for later upload, write **code: A78BP3** on the first sheet of paper with your solutions. Then hold the paper at the beginning so the code can be captured by the camera.

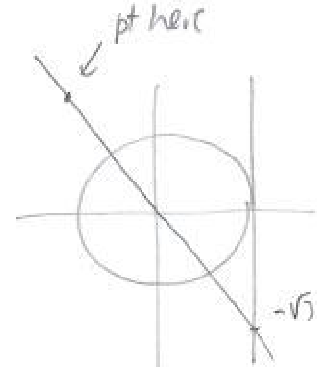
1. (12pts) Polar coordinates of two points are given.
a) Sketch the points in the plane.
b) For each point, give two additional polar coordinates, one with a positive r, one with a negative r.

$\left(3, -\frac{\pi}{3}\right)$ $\left(3, \frac{5\pi}{3}\right), \left(-3, \frac{2\pi}{3}\right)$

$\left(-2, \frac{4\pi}{5}\right)$ $\left(-2, -\frac{6\pi}{5}\right), \left(2, -\frac{\pi}{5}\right)$



2. (10pts) Convert (a picture may help):
a) $\left(7, -\frac{\pi}{4}\right)$ from polar to rectangular coordinates
b) $(-2, 2\sqrt{3})$ from rectangular to polar coordinates



a) $x = 7 \cdot \cos\left(-\frac{\pi}{4}\right) = \frac{7\sqrt{2}}{2}$

$y = 7\sin\left(-\frac{\pi}{4}\right) = -\frac{7\sqrt{2}}{2}$

b) $r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$

$= \sqrt{4 + 12}$

$= 4$

$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$

$\theta = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$ correct quadrant

$\left(4, \frac{2\pi}{3}\right)$

3. (12pts) Find the equation of the tangent line to the parametric curve $x = t^3 - 4t^2$, $y = t^3 + 5t$ at the point when $t = 1$.

$x' = 3t^2 - 8t$
$y' = 3t^2 + 5$

$x(1) = -3$ $\quad x'(1) = -5$
$y(1) = 6$ $\quad y'(1) = 8$

$m = -\frac{8}{5}$

Eq. of tan. line:

$y - 6 = -\frac{8}{5}(x-(-3))$

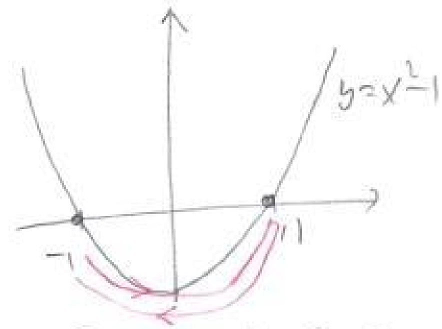
$y = -\frac{8}{5}x - \frac{24}{5} + 6$

$y = -\frac{8}{5}x + \frac{6}{5}$

4. (12pts) A particle moves along the path with parametric equations $x(t) = \cos t$, $y(t) = -\sin^2 t$ for $-\pi \le t \le \pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = \cos t$
$y = -\sin^2 t = -(1-\cos^2 t) = -(1-x^2) = x^2 - 1$

t	x	y
$-\pi$	-1	0
0	1	0
π	-1	0

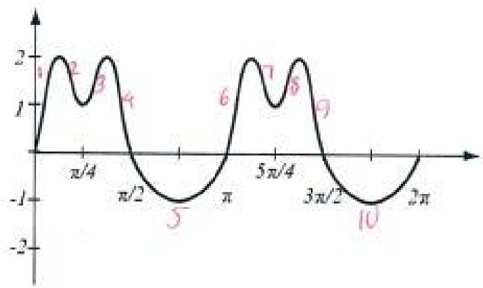


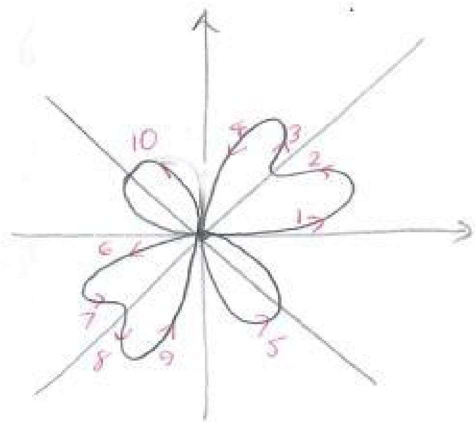
Goes from $(-1,0)$ to $(1,0)$ and back along curve $y = x^2 - 1$

5. (8pts) Find the length of the parametric curve given by $x(t) = 1 + 2t$, $y(t) = 3 - t$, $0 \le t \le 1$.

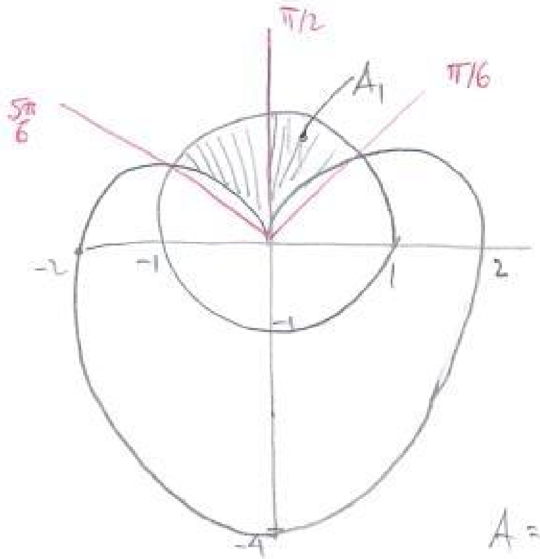
$\ell = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^1 \sqrt{2^2 + (-1)^2}\, dt = \int_0^1 \sqrt{5}\, dt = \sqrt{5}$

6. (12pts) The graph of $r = f(\theta)$ is given in cartesian coordinates. Use its intervals of increase and decrease to help you sketch the polar curve $r = f(\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.





7. (22pts) Find the area inside the curve $r = 1$ and outside $r = 2(1 - \sin\theta)$. Draw a picture showing the area you are computing.

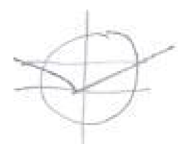


Intersection:

$1 = 2(1 - \sin\theta)$

$1 - \sin\theta = \frac{1}{2}$

$-\sin\theta = -\frac{1}{2}$



$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2A_1 = \cancel{2}\int_{\pi/6}^{\pi/2} \frac{1}{\cancel{2}}\left(1^2 - (2(1-\sin\theta))^2\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} 1 - 4(1 - 2\sin\theta + \sin^2\theta)\, d\theta = \int_{\pi/6}^{\pi/2} -3 + 8\sin\theta - 4\,\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} -5 + 8\sin\theta + 2\cos(2\theta)\, d\theta = -5\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 8\cos\theta\Big|_{\pi/6}^{\pi/2} + \frac{2\sin(2\theta)}{2}\Big|_{\pi/6}^{\pi/2}$$

$$= -\frac{5\pi}{3} - 8\left(0 - \frac{\sqrt{3}}{2}\right) + \left(0 - \frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} + 4\sqrt{3} - \frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{2} - \frac{\pi}{3}$$

8. (12pts) In a misguided attempt to fight the coronavirus, a bottle of disinfectant is thrown from the origin so that its position is given by $x(t) = 20t$, $y(t) = 18t - 5t^2$ where length is measured in meters, time in seconds.

a) Kshhhh! When does the bottle hit the ground?

b) How far did it travel from the origin?

a) Hits ground when $y(t)=0$

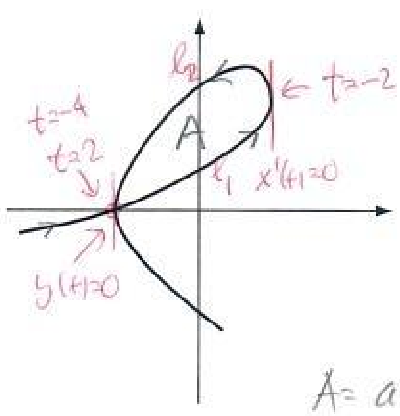
$18t - 5t^2 = 0$

$t(18-5t) = 0$

$t=0$ or $t = \frac{18}{5}$

b.) $x\left(\frac{18}{5}\right) = \overset{4}{\cancel{20}} \cdot \frac{18}{\cancel{5}} = 72$ meters

Bonus. (10pts) The graph of the parametric curve $x(t) = t^3 - 12t$, $y(t) = -t^2 - 2t + 8$ is shown. Set up the integral(s) needed to find the area enclosed by the loop.



Need points where: $x'(t)=0$ $\qquad$ $y(t)=0$

$3t^2-12=0 \qquad -t^2-2t+8=0$

$t^2-4=0 \qquad t^2+2t-8=0$

$t=\pm 2 \qquad (t+4)(t-2)=0$

$t=2,-4$

A= area between l_2 and x-axis − area between l_1 and x-axis

$$= -\int_{-2}^{2} y(t)x'(t)\,dt - \int_{-4}^{-2} y(t)x'(t)\,dt$$

$$= -\left(\int_{-4}^{-2} + \int_{-2}^{2}\right) = -\int_{-4}^{2} (-t^2-2t+8)(3t^2-12)\,dt$$