Calculus 2 — Exam 4
MAT 308, Spring 2020 — D. Ivanšić

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Show all your work!

If you are filming yourself as you take the exam for later upload, write **code: G9LK6A** on the first sheet of paper with your solutions. Then hold the paper at the beginning so the code can be captured by the camera.

Find the intervals of convergence for the series below. Don't forget to check the endpoints.

1. (16pts) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+7n}{5^n}(x-3)^n.$

$\sqrt[n]{\text{poly}(n)} \to 1$

Root test: $\displaystyle\sqrt[n]{\left|\frac{n^2+7n}{5^n}(x-3)^n\right|} = \frac{\sqrt[n]{n^2+7n}}{5}|x-3| \to \frac{1}{5}\cdot|x-3| = \frac{|x-3|}{5}$

$\dfrac{|x-3|}{5} < 1$ $\qquad |x-3| < 5$

$x=8$ get $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+7n}{5^n}\cdot 5^n = \sum_{n=0}^{\infty}(n^2+7n)$ $\qquad n^2+7n \to \infty$, diverges by divergence test

$x=-2$ get $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+7n}{5^n}(-5)^n = \sum_{n=0}^{\infty}(-1)^n(n^2+7n)$ $\qquad \lim(-1)^n(n^2+7n)$ dne, diverges by divergence test

Interval of convergence: $(-2,8)$

2. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}(x+1)^n.$

Ratio test: $\displaystyle\left|\frac{4^{n+1}}{1\cdot 3\cdot 5\cdots \underbrace{(2(n+1)-1)}_{2n+1}}(x+1)^{n+1}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n(x+1)^n}\right|$

$\displaystyle = \frac{4\cdot 1\cdot 3\cdot 5\cdots(2n-1)\,|x+1|}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} = \frac{4|x+1|}{2n+1} \to 0 < 1$

Series converges for all real x

3. (6pts) Use a known power series to find the sum:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n!} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

4. (8pts) Use a known power series to find the limit.

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{\cancel{x^4 - 5x^2}\; x^5} = \lim_{x\to 0} \frac{\cancel{x} - \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \cancel{-x} + \cancel{\frac{x^3}{6}}}{x^5} = \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5}$$

$$= \lim_{x\to 0} \frac{\cancel{x^5}\left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right)}{\cancel{x^5}} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right) = \frac{1}{5!} = \frac{1}{120}$$

5. (14pts) Use a known power series to get a power series for $\dfrac{1}{(1-3x)^2}$. State the interval of convergence (no need to check the endpoints).

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n \quad \Big| \frac{d}{dx}$$

← converges when $|3x| < 1$

$|x| < \frac{1}{3}$

$-\frac{1}{3} < x < \frac{1}{3}$

$$-\frac{1}{(1-3x)^2}\cdot(-3) = \sum_{n=0}^{\infty} 3^n \cdot n x^{n-1} \quad \Big| \div 3$$

$$\frac{1}{(1-3x)^2} = \sum_{n=0}^{\infty} 3^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} n \cdot 3^{n-1} \cdot x^{n-1}$$

6. (14pts) Use the binomial series (expand the binomial coefficients, and simplify) to write the power series expansion of the function.

$$\sqrt[4]{1+x} = (1+x)^{\frac{1}{4}} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} x^n = \sum_{n=0}^{\infty} \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\ldots\left(\frac{1}{4}-(n-1)\right)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{\frac{1}{4^n}\cdot 1\cdot(1-4)(1-2\cdot 4)\ldots(1-(n-1)\cdot 4)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\cdot 1\cdot(4-1)(4\cdot 2-1)\ldots(4(n-1)-1)}{4^n n!} x^n$$

$$= 1 + \frac{1}{4}x + \sum_{n=2}^{\infty} (-1)^{n-1} \underbrace{\frac{1\cdot 3\cdot 7\cdots(4n-5)}{4^n\cdot n!}}_{\text{formula doesn't work for } n\leq 1} x^n$$

7. (18pts) Let $f(x) = \cos x$.

a) Find the 3rd Taylor polynomial for f centered at $a = \frac{\pi}{4}$.

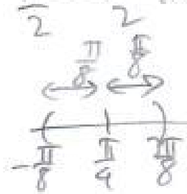
b) Use Taylor's formula to get an estimate of the error $|R_3|$ on the interval $\left(\frac{\pi}{8}, \frac{3\pi}{8}\right)$. Leave your answer as a fraction.

a)

n	$f^{(n)}$	$f^{(n)}\left(\frac{\pi}{4}\right)$
0	$\cos x$	$\sqrt{2}/2$
1	$-\sin x$	$-\sqrt{2}/2$
2	$-\cos x$	$-\sqrt{2}/2$
3	$\sin x$	$\sqrt{2}/2$
4	$\cos x$	$\sqrt{2}/2$

$$T_3(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2\cdot 1!}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{2\cdot 2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{2\cdot 3!}\left(x-\frac{\pi}{4}\right)^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$



b) $$|R_4(x)| = \left|\frac{\cos z}{4!}\left(x-\frac{\pi}{4}\right)^4\right| \leq \frac{1}{4!}\cdot \underbrace{\left|x-\frac{\pi}{4}\right|}_{\leq \frac{\pi}{8}}{}^4 = \frac{1}{24}\cdot\left(\frac{\pi}{8}\right)^4 = \frac{\pi^4}{24\cdot 8^4}$$

8. (14pts) Use the known power series for $\ln(1+x)$ to give an estimate of $\ln 1.5$ with accuracy 10^{-2}. Write the estimate as a sum (you do not have to simplify it).

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+\tfrac{1}{2}) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n \cdot 2^n} \leftarrow \text{decreasing terms}$$

Alt series estimate: need $\frac{1}{n 2^n} < 10^{-2}$

$n 2^n > 100$

n	$n 2^n$
3	$3 \cdot 8 = 24$
4	$4 \cdot 16 = 64$
5	$5 \cdot 32 = 160$ ← ignore this one

$$S_4 = \frac{1}{2} - \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4}$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{24} - \frac{1}{64}$$

estimate $\ln 1.5$ with accuracy 10^{-2}

Bonus (10pts) Find the Maclaurin series for $\arcsin x$. (Hint: what is the derivative of this function?)

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (x^2)^n = \sum_{n=0}^{\infty} \frac{-\frac{1}{2}(-\frac{1}{2}-1) \cdots (-\frac{1}{2}-(n-1))}{n!} (-x^2)^n$$

binomial series for $(1-x^2)^{-\frac{1}{2}}$

$$= \sum_{n=0}^{\infty} \frac{-1(-1-2)(-1-2\cdot 2) \cdots (-1-2(n-1))}{2^n \cdot n!} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot (2+1)(2 \cdot 2+1) \cdots (2(n-1)+1)}{2^n n!} (-1)^n x^{2n}$$

← doesn't work for $n=0$

$$(\arcsin x)' = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n n!} x^{2n}$$

$\int$ integrate

$$\arcsin x = C + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n \cdot n!} \frac{x^{2n+1}}{2n+1} = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{(2n+1) \cdot 2^n \cdot n!} x^{2n+1}$$

For $x=0$, $\arcsin 0 = C + 0$

$C = 0$