Calculus 2 — Exam 3
MAT 308, Spring 2020 — D. Ivanšić

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Show all your work!

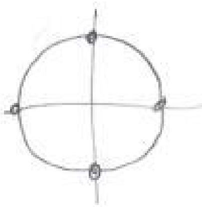
Find the limits, if they exist.

1. (6pts) $\lim\limits_{n\to\infty} \dfrac{4^{n-1}}{3^{2n}} = \lim\limits_{n\to\infty} \dfrac{4^n \cdot 4^{-1}}{(3^2)^n} = \lim\limits_{n\to\infty} \left(\dfrac{4}{3^2}\right)^n \cdot \dfrac{1}{4} = 0 \cdot \dfrac{1}{4} = 0$

$\frac{4}{9}$

$\left|\frac{4}{9}\right| < 1$

2. (6pts) $\lim\limits_{n\to\infty} \cos\dfrac{n\pi}{2} =$ does not exist, since it does not approach any single number



$\cos\frac{n\pi}{2} = 0, -1, 0, 1, 0, -1, \ldots$

$n=1$

3. (10pts) Find the limit. Use the theorem that rhymes with what a person might do, if an irritant enters their nose.

$\lim\limits_{n\to\infty} \dfrac{\sin n + 2\cos n}{3n-7}$

$-1 \le \sin n \le 1$

$-1 \le \cos n \le 1 \quad |\cdot 2$

$-3 \le \sin n + 2\cos n \le 3$

$-\dfrac{3}{3n-7} \le \dfrac{\sin n + 2\cos n}{3n-7} \le \dfrac{3}{3n-7}$

$\lim\limits_{n\to\infty} \dfrac{-3}{3n-7} = -\dfrac{3}{\infty} = 0$

$\lim\limits_{n\to\infty} \dfrac{3}{3n-7} = \dfrac{3}{\infty} = 0$

By the squeeze theorem,

$\lim\limits_{n\to\infty} \dfrac{\sin n + 2\cos n}{3n-7} = 0$

4. (6pts) Write the series using summation notation:

$$\frac{9}{1}-\frac{27}{1\cdot 2}+\frac{81}{1\cdot 2\cdot 3\cdot 4}-\frac{243}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\cdots = \frac{3^2}{0!}-\frac{3^3}{2!}+\frac{3^4}{4!}-\frac{3^5}{6!}+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{3^{n+3}}{(2n)!}$$

5. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=2}^{\infty}(-1)^n\frac{5^{n-1}}{3^{2n+1}} = \sum_{n=2}^{\infty}\frac{(-1)^n 5^n\cdot 5^{-1}}{3^{2n}\cdot 3^1} = \sum_{n=2}^{\infty}\frac{1}{15}\left(-\frac{5}{3^2}\right)^n$$

$r=-\frac{5}{9}$ converges, since $|r|<1$

$$=\frac{5^1}{3^5}\cdot\frac{1}{1-\left(-\frac{5}{9}\right)}=\frac{5}{3^5+5\cdot 3^3}=\frac{5}{378}$$

$243+5\cdot 27$

$243+135$

Determine whether the following series converge and justify your answer.

6. (6pts) $\sum_{n=1}^{\infty} e^{\frac{1}{n}}$ diverges using divergence test, since

$$\lim_{n\to\infty} e^{\frac{1}{n}} = 1 \neq 0$$

7. (12pts) $\sum_{n=1}^{\infty}\frac{\sqrt{n}+3}{n^2-2n-3}$

$$\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n}+3}{n^2-2n-3}}{\frac{1}{n^{3/2}}}=\lim_{n\to\infty}\frac{\frac{\sqrt{n}\left(1+\frac{3}{\sqrt{n}}\right)}{n^2\left(1-\frac{2}{n}-\frac{3}{n^2}\right)}}{\frac{1}{n^{3/2}}}=\frac{1+\frac{3}{\infty}}{1-\frac{2}{\infty}-\frac{3}{\infty}}=\frac{1}{1}\neq 0,\infty$$

Since $\sum\frac{1}{n^{3/2}}$ converges, so does our series, due to limit comparison test.

8. (22pts) Consider the alternating series $\sum_{n=2}^{\infty}(-1)^n \frac{n^2}{n^3+7}$.

a) Show that the sequence $\frac{n^2}{n^3+7}$ is decreasing from some point on.
b) Show the limit of the sequence in a) is 0.
c) Is the series convergent?
d) Is the series absolutely convergent? Use the integral test.

$\geq 0 \quad \leq 0$ for $x \geq 3$

a) $f(x) = \frac{x^2}{x^3+7}$ $\quad f'(x) = \frac{2x(x^3+7) - x^2 \cdot 3x^2}{(x^3+7)^2} = \frac{14x - x^4}{(x^3+7)^2} = \frac{x(14-x^3)}{(x^3+7)^2} \leq 0$ for $x \geq 3$

so sequence decreases for $n \geq 3$

b) $\lim_{n\to\infty} \frac{n^2}{n^3+7} = \lim_{n\to\infty} \frac{n^2}{n^3(1+\frac{7}{n^3})} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{1+\frac{7}{n^3}} = 0 \cdot \frac{1}{1+0} = 0$

c) Using alternating series test, the series is convergent.

d) Consider $\sum \frac{n^2}{n^3+7}$

$\int_2^{\infty} \frac{x^2}{x^3+7}\,dx = \lim_{t\to\infty} \int_2^t \frac{x^2}{x^3+7}\,dx = \left[\begin{array}{ll} u = x^3+7 & x=t,\ u=t^3+7 \\ du = 3x^2 dx & x=2,\ u=15 \\ \frac{1}{3}du = x^2 dx & \end{array}\right] = \lim_{t\to\infty} \int_{15}^{t^3+7} \frac{1}{t}\,dt$

$= \lim_{t\to\infty} \ln|t^3+7| - \ln|15|$

Determine whether the following series converge using the root or ratio test. $= \ln\infty - \ln 15 = \infty$

9. (10pts) $\sum_{n=3}^{\infty} \frac{5^{3n}}{(n+1)!}$

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{5^{3(n+1)}}{(n+2)!}}{\frac{5^{3n}}{(n+1)!}} = \frac{5^{3n+3}}{(n+2)!} \cdot \frac{(n+1)!}{5^{3n}}$

$= \frac{\cancel{5^{3n}} \cdot 5^3}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{5^{3n}}} = \frac{125}{n+2} \to 0 < 1$

Series converges absolutely using the ratio test

10. (10pts) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(\arctan n)^n}{n^2+4n}$

Root test: $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{(\arctan n)^n}{n^2+4n}\right|}$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{(\arctan n)^n}}{\sqrt[n]{n^2+4n}} = \lim_{n\to\infty}\frac{\arctan n}{\sqrt[n]{n^2+4n}} = \frac{\frac{\pi}{2}}{1} = 1.53.. > 1$$

By root test, series diverges

Bonus. (10pts) Does $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+3^n}{4^n+5^n}$ converge? (Hint: root test and dominant terms.)

Root test $\displaystyle\lim_{n\to\infty}\sqrt[n]{\left|\frac{2^n+3^n}{4^n+5^n}\right|} = \lim_{n\to\infty}\sqrt[n]{\frac{3^n\left(\frac{2^n}{3^n}+1\right)}{5^n\left(\frac{4^n}{5^n}+1\right)}} = \lim_{n\to\infty}\frac{3\sqrt[n]{\left(\frac{2}{3}\right)^n+1}}{5\sqrt[n]{\left(\frac{4}{5}\right)^n+1}}$

$$= \lim_{n\to\infty}\frac{3\left(\left(\frac{2}{3}\right)^n+1\right)^{\frac{1}{n}}}{5\left(\left(\frac{4}{5}\right)^n+1\right)^{\frac{1}{n}}} = \frac{3(0+1)^{\frac{1}{\infty}}}{5(0+1)^{\frac{1}{\infty}}} = \frac{3\cdot 1^0}{5\cdot 1^0} = \frac{3}{5} < 1$$

converges absolutely by root test