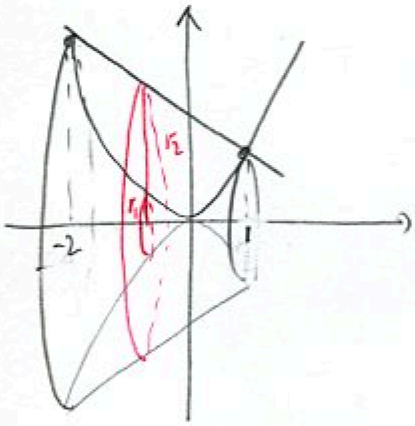
Calculus 2 — Exam 2 **MAT 308, Spring 2020 — D. Ivanšić**	**Name:** Saul Ocean *Show all your work!*

1. (24pts) The region bounded by the curves $y = x^2$ and $y = 2-x$ is rotated around the x-axis.

a) Sketch the solid and a typical cross-sectional washer.

b) Set up the integral for the volume of the solid.

c) Evaluate the integral.



b) $V = \int_{-2}^{1} A(x)\,dx = \int_{-2}^{1} \pi r_2^2 - \pi r_1^2\,dx$

$= \pi\int_{-2}^{1} (2-x)^2 - (x^2)^2\,dx = \pi\int_{-2}^{1} 4-4x+x^2-x^4\,dx$

c) $= \pi\left(4(1-(-2)) + \left(-2x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-2}^{1}\right)$

$= \pi\left(4\cdot 3 - 2(1-4) + \frac{1}{3}(1-(-8)) - \frac{1}{5}(1-(-32))\right)$

$= \pi\left(12+6+3-\frac{33}{5}\right) = \pi\left(21-\frac{33}{5}\right) = \pi\,\frac{105-33}{5}$

$= \frac{72\pi}{5}$

Intersections: $x^2 = 2-x$

$x^2+x-2=0$

$(x+2)(x-1)=0$

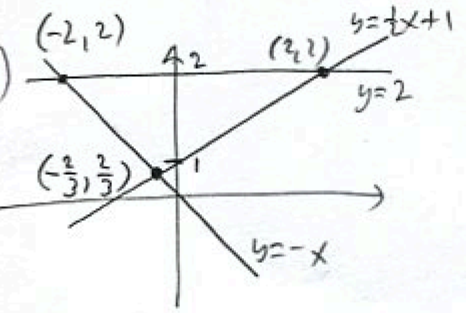
$x=-2,1$

2. (14pts) Consider the triangle bounded by lines $y = \frac{1}{2}x + 1$ and $y = -x$ and $y = 2$.

a) Sketch the triangle.

b) Set up the integral that computes its area. Simplify, but do not evaluate the integral.

a)



b) Area easier in y-variable:

$y=\frac{1}{2}x+1$ $\quad$ $y=-x$

$x=2y-2$ $\quad$ $x=-y$

$\int_{2/3}^{2} 2y-2-(-y)\,dy = \int_{2/3}^{2} 3y-2\,dy$

Intersection $\frac{1}{2}x+1=-x$

$\frac{3}{2}x=-1$

$x=-\frac{2}{3},\ y=\frac{2}{3}$

$\frac{1}{2}x+1=2$

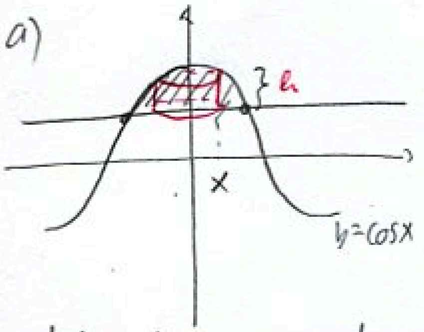
$\frac{1}{2}x=1 \quad x=2$

Done in variable x requires two integrals:

$\int_{-2}^{-\frac{2}{3}}(2-(-x))\,dx + \int_{-2/3}^{2} 2-\left(\frac{1}{2}x+1\right)dx = \int_{-2}^{-\frac{2}{3}} x+2\,dx + \int_{-\frac{2}{3}}^{2} 1-\frac{1}{2}x\,dx$

3. (16pts) There are infinitely many regions that are above line $y = \frac{1}{2}$ and below the curve $y = \cos x$. Rotate the region that intersects the y-axis about the y-axis to get a solid.
a) Sketch the solid and a typical cylindrical shell.
b) Set up the integral for the volume of the solid using the shell method. Simplify, but do not evaluate the integral.

a)



Intersection: $\cos x = \frac{1}{2}$

$x = -\frac{\pi}{3}, \frac{\pi}{3}$

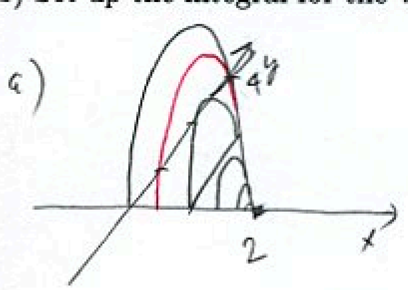
b) $V = \int_0^{\pi/3} S(x)\,dx = \int_0^{\pi/3} 2\pi r \cdot h\,dx$

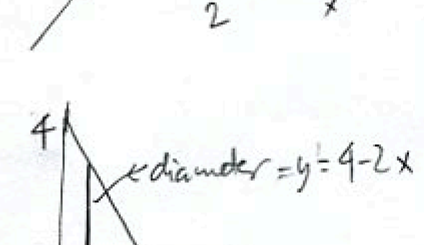
$= \int_0^{\pi/3} 2\pi \cdot x \cdot \left(\cos x - \frac{1}{2}\right) dx$

$= \int_0^{\pi/3} \pi x \,(2\cos x - 1)\,dx$

4. (16pts) The base of a solid is the triangle in the xy-plane with vertices $A = (0,0)$, $B = (2,0)$ and $C = (0,4)$. The cross-sections of the solid perpendicular to the x-axis are half-disks whose diameters lie in the triangle.
a) Sketch the solid and a typical cross-section.
b) Set up the integral for the volume of the solid. Simplify, but do not evaluate the integral.

a)





line is $y = 4 - 2x$

$r = \frac{1}{2}(4-2x) = 2-x$

b) $V = \int_0^2 A(x)\,dx = \int_0^2 \frac{1}{2}\pi r^2\,dx$

$= \int_0^2 \frac{1}{2}\pi (2-x)^2\,dx$

$= \frac{\pi}{2}\int_0^2 (x-2)^2\,dx$

5. (14pts) Compute the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{1}{2}}$ from $x = 1$ to $x = 4$.

$$y' = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{4}x^{-\frac{1}{2}}$$

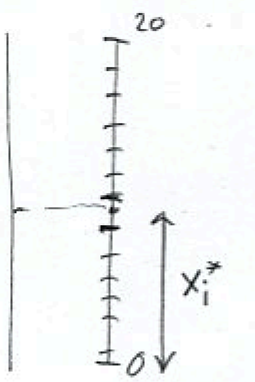
$$= \sqrt{x} + \frac{1}{4\sqrt{x}}$$

$$\text{length} = \int_1^4 \sqrt{1+\left(\sqrt{x}-\frac{1}{4\sqrt{x}}\right)^2}\,dx = \int_1^4 \sqrt{1+x-\frac{1}{2}+\frac{1}{16x}}\,dx = \int_1^4 \sqrt{x+\frac{1}{2}+\frac{1}{16x}}$$

$$= \int_1^4 \sqrt{\left(\sqrt{x}+\frac{1}{4\sqrt{x}}\right)^2}\,dx = \int_1^4 x^{\frac{1}{2}}+\frac{1}{4}x^{-\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}+\frac{1}{4}\cdot 2\cdot x^{\frac{1}{2}}\Big|_1^4$$

$$= \frac{2}{3}(8-1)+\frac{1}{2}(2-1) = \frac{14}{3}+\frac{1}{2} = \frac{31}{6}$$

6. (16pts) A leaky bucket is lifted from a well with depth 20 meters to the surface. The bucket weighs 1kg, starts with 10 liters of water at bottom and has only 2 liters by the time it is pulled to the top (assume it empties at a constant rate and rope weight is negligible). Set up the integral for the work needed to lift the bucket from the bottom of the well to the top. Assume $g = 10$ and water density = 1kg/liter. Simplify, but do not evaluate the integral.



Loses 8 liters over 20 meters

$\frac{8}{20}$ l/m = $\frac{2}{5}$ l/m

Volume of water in bucket at height x_i^*:

$10 - \frac{2}{5}x_i^*$, mass of bucket with water:

$1 + (10 - \frac{2}{5}x_i^*)\cdot \underset{\text{density}}{1} = 11 - \frac{2}{5}x_i^*$

Small amount of work is $(11 - \frac{2}{5}x_i^*)\cdot g \cdot \Delta x$

$$W = \sum_{i=1}^{n} 10\left(11 - \frac{2}{5}x_i^*\right)\Delta x$$

$$W = \int_0^{20} 10\left(11 - \frac{2}{5}x\right)dx$$

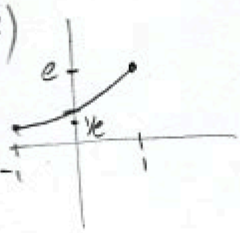
$$= \int_0^{20} 110 - 4x\,dx$$

Bonus (10pts) Consider the surface obtained by rotating the curve $y = e^x$, $-1 \le x \le 1$, around the x-axis.

a) Set up the integral for surface area in variable x.

b) Set up the integral for surface area in variable y.

c) Do not evaluate the integrals, but verify that they are equal.

a) 

$$\int_{-1}^{1} 2\pi r\,ds = \int_{-1}^{1} 2\pi e^x \sqrt{1+(e^x)^2}\,dx = 2\pi \int_{-1}^{1} e^x \sqrt{1+e^{2x}}\,dx$$

$y=e^x,\ y'=e^x$

b) $x = \ln y$

$x' = \frac{1}{y}$

$$\int_{1/e}^{e} 2\pi r\,ds = \int_{1/e}^{e} 2\pi y \sqrt{1+\left(\tfrac{1}{y}\right)^2}\,dy = \int_{1/e}^{e} 2\pi \sqrt{y^2\left(1+\tfrac{1}{y^2}\right)}\,dy$$

$$= \int_{1/e}^{e} 2\pi \sqrt{y^2+1}\,dy$$

c) $$\int_{1/e}^{e} \sqrt{y^2+1}\,dy = \begin{bmatrix} y=e^x & y=e,\ x=1 \\ dy=e^x dx & y=e,\ x=-1 \end{bmatrix} = \int_{-1}^{1} \sqrt{(e^x)^2+1}\cdot e^x dx = \int_{-1}^{1} e^x \sqrt{1+e^{2x}}\,dx$$

(2π is same in both integrals