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| **Calculus 2 — Exam 1** | |
| **MAT 308, Spring 2020 — D. Ivanšić** | |

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*Show all your work!*

Find the following integrals:

**1.** (6pts) $\int x\cos(3x)\,dx = \begin{bmatrix} u=x & dv=\cos(3x)\,dx \\ du=dx & v=\frac{\sin(3x)}{3}\end{bmatrix} = \frac{x\sin(3x)}{3} - \int \frac{\sin(3x)}{3}\,dx$

$$= \frac{x\sin(3x)}{3} + \frac{\cos(3x)}{9} + C$$

**2.** (9pts) $\int \sin^3 x\cos^3 x\,dx = \int \sin^3 x\underbrace{\cos^2 x}_{=1-u^2}\cos x\,dx = \begin{bmatrix} u=\sin x \\ du=\cos x\,dx\end{bmatrix}$

$$= \int u^3(1-u^2)\,du = \int u^3-u^5\,du = \frac{u^4}{4}-\frac{u^6}{6} = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$$

Determine whether the following improper integrals converge, and, if so, evaluate them.

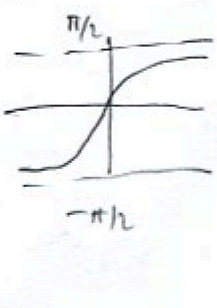
**3.** (8pts) $\int_1^\infty \frac{x+1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \underbrace{\frac{1}{x^2}}_{x^{-2}}+\underbrace{\frac{1}{x^3}}_{x^{-3}}\,dx = \lim_{t\to\infty}\left(\frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}\right)\Big|_1^t$

$$= \lim_{t\to\infty}\left(-\frac{1}{x}-\frac{1}{2x^2}\right)\Big|_1^t = \lim_{t\to\infty} \underbrace{-\frac{1}{t}}_{\to 0} - \frac{1}{2t^2} - \left(-1-\frac{1}{2}\right) = \underbrace{-\frac{1}{\infty}-\frac{1}{\infty}}_{=0}+\frac{3}{2} = \frac{3}{2}$$

converges

**4.** (8pts) $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan x\Big|_0^t$

$$= \lim_{t\to\infty} \arctan t - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$ converges



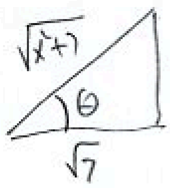
Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

**5.** (12pts) $\int \frac{x^3}{\sqrt{x^2+7}}\,dx = \begin{bmatrix} x=\sqrt7\tan\theta \\ dx = \sqrt7\sec^2\theta\,d\theta\end{bmatrix} = \int \frac{\sqrt7^3\tan^3\theta\cdot\sqrt7\sec^2\theta}{\sqrt{7\tan^2\theta+7}}\,d\theta$

$$= \int \frac{\sqrt7^3\cancel{\sqrt7}\tan^3\theta\sec^2\theta}{\cancel{\sqrt7}\sec\theta}\,d\theta = \int 7^{\frac32}\tan^3\theta\sec\theta\,d\theta = 7^{\frac32}\int \underbrace{\tan^2\theta}_{(\sec^2\theta-1)}\sec\theta\tan\theta\,d\theta$$

$$= \begin{bmatrix} u=\sec\theta \\ du=\sec\theta\tan\theta\end{bmatrix} = 7^{\frac32}\int(u^2-1)\,du = 7^{\frac32}\left(\frac{u^3}{3}-u\right)$$

$$= 7^{\frac32}\left(\frac{\sec^3\theta}{3}-\sec\theta\right) = \frac{7^{\frac32}}{3}\left(\frac{\sqrt{x^2+7}}{\sqrt7}\right)^3 - 7^{\frac32}\frac{\sqrt{x^2+7}}{\sqrt7} = \frac{(x^2+7)^{\frac32}}{3} - 7\sqrt{x^2+7}+C$$




$\frac{x}{\sqrt7}=\tan\theta$

$\sin\theta = \frac{\sqrt2}{2}$

$\sin\theta = 0$

**6.** (14pts) $\int_0^{\sqrt2} x^2\sqrt{4-x^2}\,dx = \begin{bmatrix} x=2\sin\theta & x=\sqrt2,\ \theta=\frac{\pi}{4} \\ dx=2\cos\theta\,d\theta & x=0,\ \theta=0\end{bmatrix}$

$$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = \int_0^{\frac{\pi}{4}} 8\sin^2\theta\cdot 2\cos\theta\cdot\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/4}\sin^2\theta\cos^2\theta\,d\theta = 16\int_0^{\frac{\pi}{4}}\left(\frac{\sin(2\theta)}{2}\right)^2 d\theta = 4\int_0^{\frac{\pi}{4}}\sin^2(2\theta)\,d\theta$$

$$= 4\int_0^{\pi/4}\frac{1-\cos(4\theta)}{2} = 2\cdot\left(\frac{\pi}{4}-0\right) - 2\frac{\sin(4\theta)}{4}\Big|_0^{\pi/4}$$

$$= \frac{\pi}{2} - \frac12(\underbrace{\sin\pi-\sin0}_{=0}) = \frac{\pi}{2}$$

Use the method of partial fractions to find the following integrals.

**7.** (14pts) $\displaystyle\int \frac{3x^3 - 5x^2 + 9x - 5}{(x^2+1)^2}\,dx =$ $\displaystyle\int \frac{3x-5}{x^2+1}\,dx + \int \frac{6x}{(x^2+1)^2}$

$\dfrac{3(x^2+1)^{-1}}{-1}$

$$\frac{3x^3-5x^2+9x-5}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \Big| \cdot (x^2+1)^2$$

$$3x^3-5x^2+9x-5 = (Ax+B)(x^2+1) + Cx + D$$

$x^0$ -- $-5 = B + D$ $\qquad A=3,\ B=-5$

$x^1$ -- $9 = A + C \Rightarrow C = 9 - A = 6$

$x^2$ -- $-5 = B$ $\qquad -5 + D = -5$

$x^3$ -- $3 = A$ $\qquad D = 0$

$$= \int \frac{3x}{x^2+1} - \int \frac{5}{x^2+1} + \int \frac{6x}{(x^2+1)^2}$$

$$= \frac{3}{2}\ln|x^2+1| - 5\arctan x - \frac{3}{x^2+1} + C$$

**8.** (9pts) Use comparison to determine whether the improper integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{x}\,dx$ converges.

$1 \geq \cos x \geq \frac{\sqrt{2}}{2}$ on $[0, \frac{\pi}{4}]$

So $\dfrac{1}{x} \geq \dfrac{\cos x}{x} \geq \dfrac{\sqrt{2}}{2x}$

$$\int_0^{\pi/4} \frac{\sqrt{2}}{2x}\,dx = \lim_{t\to 0+} \int_t^{\pi/4} \frac{\sqrt{2}}{2}\frac{1}{x}\,dx = \lim_{t\to 0+} \frac{\sqrt{2}}{2}\ln|x|\Big|_t^{\pi/4} = \lim_{t\to 0+} \frac{\sqrt{2}}{2}\Big(\ln\frac{\pi}{4} - \underbrace{\ln t}_{\to -\infty}\Big)$$

$$= \frac{\sqrt{2}}{2}\left(\ln\frac{\pi}{4} - (-\infty)\right) = \infty, \text{ so diverges}$$

**9.** (20pts) The integral $\int_0^3 e^{-x^2}\,dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $e^{-x^2}$ is not expressible using elementary functions.

a) Write the expression you would use to calculate $M_6$, the midpoint rule with 6 subintervals. All the terms need to be explicitly written, do not use $f$ in the sum.

b) Find $y''$ for $y = e^{-x^2}$.

c) The graph of $y''$ is shown: use it to find the error estimate for $M_n$ in general.

d) Estimate the error for $M_6$.

e) What should $n$ be in order for $M_n$ to give you an error less than $10^{-4}$?

a) $\frac{1}{4}\ \frac{3}{4}\ \frac{5}{4}\ \frac{7}{4}\ \frac{9}{4}\ \frac{11}{4}$

$0\ \frac{1}{2}\ 1\ \frac{3}{2}\ 2\ \frac{5}{2}\ 3$

$$M_6 = \frac{1}{2}\left(e^{-\left(\frac{1}{4}\right)^2} + e^{-\left(\frac{3}{4}\right)^2} + e^{-\left(\frac{5}{4}\right)^2} + e^{-\left(\frac{7}{4}\right)^2} + e^{-\left(\frac{9}{4}\right)^2} + e^{-\left(\frac{11}{4}\right)^2}\right)$$

b) $y = e^{-x^2}$

$y' = e^{-x^2}\cdot(-2x) = -2xe^{-x^2}$

$y'' = -2\left(1\cdot e^{-x^2} + x\cdot e^{-x^2}(-2x)\right)$

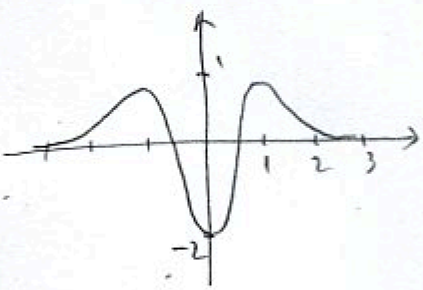
$= -2e^{-x^2}(1-2x^2)$

$= 2(2x^2-1)e^{-x^2}$

c) using the graph $K_2 = 2$ $(|y''| \le 2$ on $[0,3])$

$$|E_M| \le \frac{2\cdot(3-0)^3}{24\cdot n^2} = \frac{2\cdot 3^3}{24n^2} = \frac{2\cdot 3^2}{8n^2} = \frac{9}{4n^2}$$

d) $$|E_n| \le \frac{9}{4\cdot 6^2} = \frac{9}{4\cdot 36} = \frac{1}{16}$$




e) $$\frac{9}{4n^2} \le 10^{-4} \qquad 1\cdot 10^4$$

$$\frac{9\cdot 10^4}{4} \le n^2 \qquad n \ge \sqrt{\frac{9\cdot 10^4}{4}} = \frac{3\cdot 10^2}{2} = 150$$

**Bonus** (10pts) Find the reduction formula that reduces $\int \frac{dx}{(x^2+a^2)^n}$ to $\int \frac{dx}{(x^2+a^2)^{n-1}}$.
Start on $\int \frac{dx}{(x^2+a^2)^{n-1}}$ with an integration by parts, then rewrite an $x^2$ in the new integral as $x^2 + a^2 - a^2$ and see what you can do.

$$\int \frac{dx}{(x^2+a^2)^{n-1}} = \begin{bmatrix} u = (x^2+a^2)^{-(n-1)} & dv = 1dx \\ du = -(n-1)(x^2+a^2)^{-n}\cdot 2x & v = x \end{bmatrix} = \overbrace{\frac{x}{(x^2+a^2)^{n-1}}}^{*} + \int \frac{2(n-1)x^2}{(x^2+a^2)^n}\,dx$$

$$= * + 2(n-1)\int \frac{x^2+a^2-a^2}{(x^2+a^2)^n}\,dx = * + 2(n-1)\int \frac{x^2+a^2}{(x^2+a^2)^n} - \frac{a^2}{(x^2+a^2)^n}\,dx = * + 2(n-1)\int \frac{dx}{(x^2+a^2)^{n-1}} - 2(n-1)a^2\int \frac{dx}{(x^2+a^2)^n}$$

$$\int \frac{dx}{(x^2+a^2)^{n-1}} = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{dx}{(x^2+a^2)^{n-1}} - 2(n-1)a^2\int \frac{dx}{(x^2+a^2)^n} \quad \text{so}$$

$$2(n-1)a^2\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^{n-1}} + \left(2(n-1)-1\right)\int \frac{dx}{(x^2+a^2)^{n-1}}$$

$$\int \frac{dx}{(x^2+a^2)^n}\,dx = \frac{1}{2(n-1)a^2}\,\frac{x}{(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}}$$