Calculus 2 — Exam 0
MAT 308, Spring 2020 — D. Ivanšić

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Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dx}\left(5x^4 - u^3 + \sqrt[4]{x^7} - \frac{12}{x^5}\right) = 20x^3 + \frac{7}{4}x^{\frac{3}{4}} + 60x^{-6} = 20x^3 + \frac{7}{4}\sqrt[4]{x^3} + \frac{60}{x^6}$

constant $x^{7/4}$ x^{-5}

difference of cubes 2

2. (6pts) $\frac{d}{dt}(\sqrt[3]{t} - 1)(\sqrt[3]{t}^{\,2} + \sqrt[3]{t} + 1) = \frac{d}{dt}\left(\sqrt[3]{t}^{3} - 1^3\right) = \frac{d}{dt}(t-1) = 1$

$OR = \frac{d}{dt}(t^{\frac{1}{3}} - 1)(t^{\frac{2}{3}} + t^{\frac{1}{3}} + 1) = \frac{1}{3}t^{-\frac{2}{3}}(t^{\frac{2}{3}} + t^{\frac{1}{3}} + 1) + (t^{\frac{1}{3}} - 1)(\frac{2}{3}t^{-\frac{1}{3}} + \frac{1}{3}t^{-\frac{2}{3}})$

$= \frac{1}{3}(1 + t^{-\frac{1}{3}} + t^{-\frac{2}{3}}) + \frac{1}{3}(2 \underbrace{- 2t^{-\frac{1}{3}} + t^{-\frac{1}{3}}}_{= -t^{-1/3}} - t^{-\frac{2}{3}}) = \frac{1}{3}(1 + \cancel{t^{-1/3}} + \cancel{t^{-2/3}}) + \frac{1}{3}(2 - \cancel{t^{-1/3}} - \cancel{t^{-2/3}}) = 1$

3. (8pts) $\frac{d}{dz}\frac{4z^2 + 1}{(z-5)^2} = \longrightarrow = \frac{8z(z-5)^2 - (4z^2+1)2(z-5)}{(z-5)^4}$

$= \frac{(8z(z-5) - 2(4z^2+1))\cancel{(z-5)}}{(z-5)^{\cancel{4}\,3}} = \frac{8z^2 - 40z - 8z^2 - 2}{(z-5)^3} = -\frac{40z + 2}{(z-5)^3}$

4. (4pts) $\frac{d}{dx}\frac{1}{xe^x} = \frac{d}{dx}(xe^x)^{-1} = -(xe^x)^{-2}(1 \cdot e^x + x \cdot e^x)$

$= -\frac{e^x(x+1)}{x^2(e^x)^2} = -\frac{x+1}{x^2 e^x}$

5. (7pts) (This is a known derivative, your job is to verify it here.)

$\frac{d}{d\theta}\ln|\sec\theta + \tan\theta| = \frac{1}{\sec\theta + \tan\theta} \cdot (\sec\theta\tan\theta + \sec^2\theta)$

$= \frac{1}{\cancel{\sec\theta + \tan\theta}}\sec\theta\,(\cancel{\tan\theta + \sec\theta}) = \sec\theta$

6. (6pts) $\frac{d}{dx}\left(\sqrt{x} - \frac{3}{\sqrt{x}}\right)\ln x = \frac{d}{dx}\left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\right)\ln x$

$= \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}\right)\ln x + \left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}\right) \cdot x^{-1}$

$= \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}\right)\ln x + x^{-\frac{1}{2}} - 3x^{-\frac{3}{2}} = x^{-\frac{1}{2}}\left(\frac{1}{2}\ln x + 1\right) + x^{-\frac{3}{2}}\left(\frac{3}{2}\ln x - 3\right)$

7. (5pts) Let $f(x) = 2^{7x}$. What is $f^{(44)}(x)$, the 44th derivative of f? Justify your answer.

$y = 2^{7x}$

$y' = \ln 2 \cdot 2^{7x} \cdot 7 = 7\ln 2 \cdot 2^{7x}$

$y'' = 7\ln 2 \cdot \underbrace{7\ln 2 \cdot 2^{7x}}_{=(2^{7x})'} = (7\ln 2)^2\, 2^{7x}$

$y''' = (7\ln 2)^2 \cdot 7\ln 2 \cdot 2^{7x} = (7\ln 2)^3 \cdot 2^{7x}$

$f^{(44)}(x) = (7\ln 2)^{44} \cdot 2^{7x}$

Find the following limits. Use L'Hospital's rule if needed.

8. (2pts) $\lim\limits_{x\to\infty} \frac{1}{x^2} = \frac{1}{\infty^2} = \frac{1}{\infty} = 0$

9. (6pts) $\lim\limits_{x\to\infty} \frac{x^3 - 5x^2 + 7x + 9}{x^2 - 4x + 5} = \lim\limits_{x\to\infty} \frac{x^3\left(1 - \overset{\to 0}{\frac{5}{x}} + \overset{\to 0}{\frac{7}{x^2}} + \overset{\to 0}{\frac{9}{x^3}}\right)}{x^2\left(1 - \underset{\to 0}{\frac{4}{x}} + \underset{\to 0}{\frac{5}{x^2}}\right)} = \lim\limits_{x\to\infty} x \cdot 1 = \infty \cdot 1 = \infty$

10. (8pts) $\lim\limits_{x\to 0} (1 + 2x)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0} \ln y} = e^2$

$y = (1+2x)^{\frac{1}{x}}$

$\ln y = \frac{1}{x} \ln(1+2x)$

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \frac{\overset{\to 0}{\ln(1+2x)}}{\underset{\to 0}{x}} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{\frac{1}{1+2x} \cdot 2}{1} = \frac{2}{1+0} = 2$

Find the following antiderivatives.

11. (7pts) $\int 4x^4 - \frac{4}{\sqrt{1-x^2}} + \sqrt[3]{x^{11}} + e^3\,dx =$ $4\cdot\frac{x^5}{5} - 4\arcsin x + \frac{3}{14}x^{\frac{14}{3}} + e^3 x + C$

$x^{\frac{11}{3}}$ $\uparrow$ constant

12. (3pts) $\int \cos(4x+1)\,dx =$ $\frac{\sin(4x+1)}{4} + C$

13. (7pts) $\int \frac{x^2+1}{\sqrt{x}}\,dx =$ $\int \frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\,dx = \int x^{\frac{3}{2}} + x^{-\frac{1}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$

$x^{1/2}$

Use the substitution rule in the following integrals:

14. (7pts) $\int \frac{4x-3}{(4x^2-6x+5)^3}\,dx =$ $\left[\begin{array}{l} u = 4x^2-6x+5 \\ du = 8x-6 = 2(4x-3)\,dx \\ \frac{du}{2} = (4x-3)\,dx \end{array}\right]$

$= \int \frac{1}{u^3}\,\frac{1}{2}du$

$= \frac{u^{-2}}{-2}\cdot\frac{1}{2} = -\frac{1}{4u^2} = -\frac{1}{4(4x^2-6x+5)^2} + C$

15. (10pts) $\int_0^{\frac{2\pi}{3}} \frac{\sin x}{1+\cos^2 x}\,dx =$ $\left[\begin{array}{ll} u = \cos x & x = \frac{2\pi}{3},\ u = \cos\frac{2\pi}{3} = -\frac{1}{2} \\ du = -\sin x\,dx & x = 0,\ u = \cos 0 = 1 \end{array}\right]$

$= \int_1^{-1/2} \frac{-du}{1+u^2} = \int_{-1/2}^{1} \frac{1}{1+u^2}\,du = \arctan u\Big|_{-1/2}^{1} = \arctan 1 - \arctan(-\tfrac{1}{2})$

$= \frac{\pi}{4} + \arctan\left(\frac{1}{2}\right)$

$(x+5)(x-3)$

16. (8pts) Find the equation of the tangent line to the curve $y = x^2 + 2x - 15$ at the point $(2, -7)$. Sketch the curve and the tangent line on the same graph.

$y = x^2 + 2x - 15$

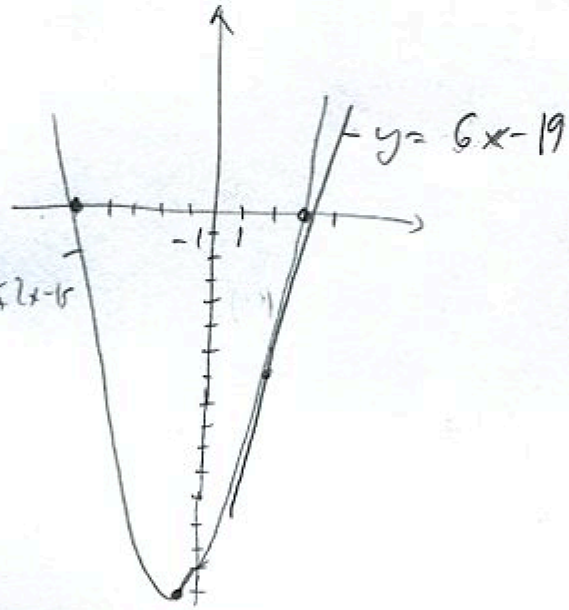
$y' = 2x + 2$

Equation of tan. line:

$m = 2 \cdot 2 + 2 = 6$

$y - (-7) = 6(x-2)$

$\boxed{y = 6x - 19}$



Bonus. (10pts) The rear inside cover of our book claims that

$$\int x^2 \sqrt{a^2 - x^2}\, dx = \underbrace{\frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2}}_{\text{double product rule}} + \frac{a^4}{8} \arcsin \frac{x}{a} + C$$

Verify this formula by differentiating.

$\frac{d}{dx}$

$$= \frac{1}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{x}{8}\left(4x\sqrt{a^2-x^2} + (2x^2 - a^2)\frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2-x^2}}\right) + \frac{a^4}{8}\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{1}{8}(2x^2 - a^2)\left(\sqrt{a^2-x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right) + \frac{x^2}{2}\sqrt{a^2-x^2} + \frac{a^4}{8} \cdot \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{1}{8}(2x^2 - a^2)\frac{a^2 - x^2 - x^2}{\sqrt{a^2-x^2}} + \frac{x^2}{2}\sqrt{a^2-x^2} + \frac{a^4}{8\sqrt{a^2-x^2}}$$

$$= \frac{(2x^2 - a^2)(a^2 - 2x^2)}{8\sqrt{a^2-x^2}} + \frac{4x^2(a^2-x^2) + a^4}{8\sqrt{a^2-x^2}} = \frac{-(4x^4 - 4x^2a^2 + a^4) + 4x^2a^2 - 4x^4 + a^4}{8\sqrt{a^2-x^2}}$$

$-(2x^2-a^2)^2$

$$= \frac{8x^2a^2 - 8x^4}{8\sqrt{a^2-x^2}} = \frac{8x^2(a^2-x^2)}{8\sqrt{a^2-x^2}} = x^2\sqrt{a^2-x^2}$$