10.5 Area and length in polar coordinates

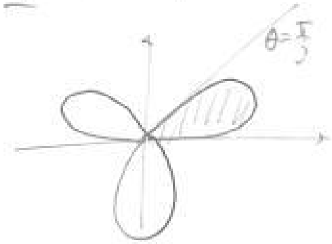
How to find area

Subdivide into wedges. Each has area $\frac{1}{2} f(\theta_i^*)^2 \Delta\theta$, so area is approx.

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\left(f(\theta_i^*)\right)^2 \Delta\theta \qquad \Big/ \lim_{n\to\infty}$$

$$\boxed{A = \int_a^b \frac{1}{2}\left(f(\theta)\right)^2 d\theta}$$

Ex. Area of the rose with 3 petals $r = \sin 3\theta$

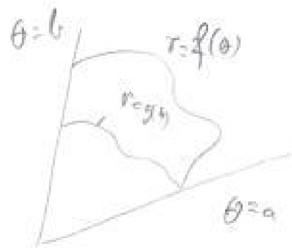


$$A = 3\cdot\int_0^{\pi/3} \sin^2 3\theta \, d\theta$$

$$= 3\cdot\int_0^{\pi/3} \frac{1-\cos 6\theta}{2} \, d\theta$$

$$= \frac{3}{2}\cdot\frac{\pi}{3} = \frac{\pi}{2}$$

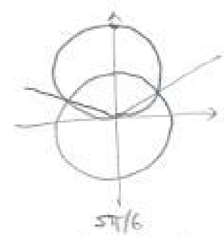
Area between two polar curves:



$$A = \frac{1}{2}\int_a^b \left(f(\theta)\right)^2 - \left(g(\theta)\right)^2 d\theta$$

since $A = A_1 - A_2$

Ex: Area inside $r = \sin\theta$, outside $r = \frac{1}{2}$



$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\sin^2\theta - \frac{1}{4}\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \frac{1-\cos 2\theta}{2} - \frac{1}{4} \, d\theta$$

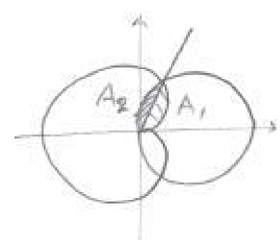
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \frac{1}{4} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left(\frac{1}{4}\frac{2\pi}{3} - \frac{1}{4}\sin 2\theta \Big|_{\pi/6}^{5\pi/6}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{4}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{4}$$

Ex. Setting integral for area inside both $r = 1-\cos\theta$ and $r = \cos\theta$



$$1-\cos\theta = \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

$$A_1 = \int_0^{\pi/3} \frac{1}{2}\left(1-\cos\theta\right)^2$$

$$A = 2(A_1 + A_2)$$

$$A_2 = \int_{\pi/3}^{\pi/2} \frac{1}{2}\cos^2\theta \, d\theta$$

Length of polar curve $r = f(\theta)$ is

$$\ell = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Use: $x = r(\theta)\cos\theta$

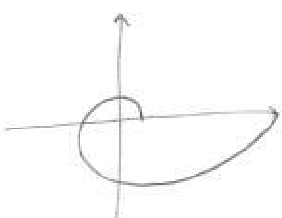
$y = r(\theta)\sin\theta$

$x' = r'\cos\theta - r\sin\theta$

$y' = r'\sin\theta + r\cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + (r')^2$$

Ex: Find the length of $r = e^\theta$, $\theta \in 0, 2\pi$



$$\ell = \int_0^{2\pi} \sqrt{(e^\theta)^2 + (e^\theta)^2}\, d\theta$$

$$= \sqrt{2}\,(e^{2\pi} - 1)$$