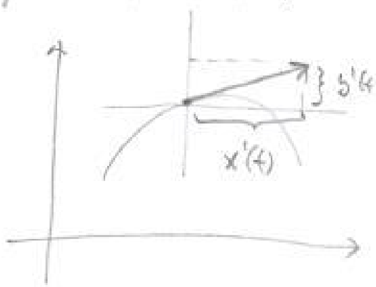
10.2 Tangents and areas

$x = x(t)$
$y = y(t)$

Suppose a particle is moving along a parametric curve. At every moment the particle has a velocity that is a vector tangent to the curve.



Slope of tangent line is $\frac{\text{rise}}{\text{run}} = \frac{y'(t)}{x'(t)}$.

(If $x'(t)=0$ and $y'(t) \neq 0$, tangent is vertical)

Ex: Find the equation of the tangent line to $x = 1-t^2$, $y = t^3 - t$ at point $\left(\frac{3}{4}, -\frac{3}{8}\right)$

$1 - t^2 = \frac{3}{4}$ $\quad$ $t^3 - t = -\frac{3}{8}$ $\quad$ So $t = \frac{1}{2}$

$t = \pm\frac{1}{2}$ $\quad$ $t = \frac{1}{2}$

$x' = -2t$

$y' = 3t^2 - 1$

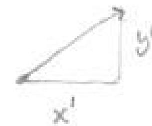
$$m = \frac{y'(\frac{1}{2})}{x'(\frac{1}{2})} = \frac{\frac{3}{4} - 1}{-1} = \frac{1}{4}$$

$$y + \frac{3}{8} = \frac{1}{4}\left(x - \frac{3}{4}\right)$$

$$y = \frac{1}{4}x - \frac{3}{16} + \frac{3}{8}$$

$$y = \frac{1}{4}x + \frac{3}{16}$$

We can use places where tangent line is horizontal or vertical to sketch a curve.



vert. if $x' = 0$

horiz. if $y' = 0$

Ex. Sketch the curve $x = 1 - t^2$

$y = t^3 - t = t(t^2 - 1)$

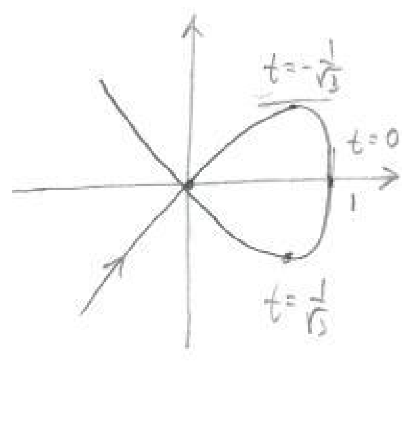
Use critical pts of x' and y'

$x' = -2t$ $\quad$ $t = 0$ $\quad$ $3t^2 - 1 = 0$

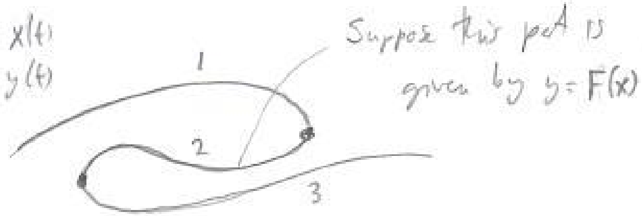
$y' = 3t^2 - 1$ $\quad$ $t = \pm\frac{1}{\sqrt{3}}$

		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$	
x'	+		+	0	−		−
y'	+	0	−		−	0	−
x	→		→		←		←
y	↑		↓		↓		↑
curve	↗		↘		↙		↖

t	x	y
$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$	$\frac{2}{3\sqrt{3}}$
0	1	0
$\frac{1}{\sqrt{3}}$	$\frac{2}{3}$	$-\frac{2}{3\sqrt{3}}$



$x = x(t)$
$y = y(t)$



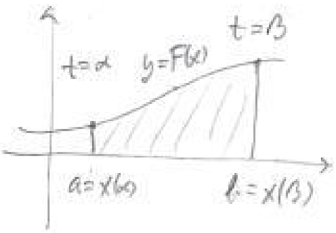
A parametric curve is not a function, but can be broken up into pieces, each of which is the graph of a function.

We can use this to again find equation for slope of tangent line:

$$y = F(x)$$
$$y(t) = F(x(t)) \quad \Big/ \frac{d}{dt}$$
$$y'(t) = F'(x(t)) \cdot x'(t)$$
$$F'(x) = \frac{y'(t)}{x'(t)}$$

May also write as $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ ← formally expand the fraction.

Area under parametric curve:



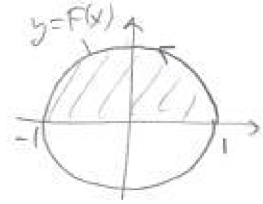
$$\boxed{A = \int_\alpha^\beta y(t)\,x'(t)\,dt.}$$

$$A = \int_a^b F(x)\,dx = \begin{bmatrix} x = x(t) & x=b, \ t=\beta \\ dx = x'(t)\,dt & x=a, \ t=\alpha \end{bmatrix}$$

$$= \int_\alpha^\beta F(x(t))\,x'(t)\,dt = \int_\alpha^\beta y(t)\,x'(t)\,dt$$

$$=$$

Ex. Area of disk of radius r



$x = r\cos t$
$y = r\sin t$ $\quad t \in [0, \pi]$

$x' = -r\sin t$

$$A = \int_0^\pi r\sin t\,(-r\sin t)\,dt$$

$$= -r^2 \int_0^\pi \sin^2 t\,dt = -r^2 \int_0^\pi \frac{1-\cos 2t}{2}\,dt$$

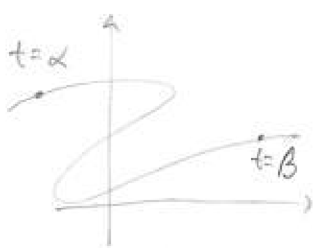
$$= -r^2 \cdot \tfrac{1}{2}\pi$$

got negative because as t increases we move along curve from right to left, which is same as doing $\int_1^{-1} F(x)\,dx$.

So $A = \int_\alpha^\beta y(t)x'(t)\,dt$ provided we move from left to right along curve

$A = \int_\beta^\alpha y(t)\,x'(t)\,dt$ otherwise

10.3 Arc length and surface area

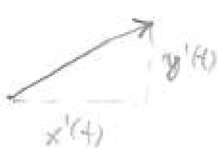


$x = x(t)$
$y = y(t)$

Length of a parametric curve is given by

$$l = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note:



$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is the length of the tangent vector

Ex: Circumference of a circle of radius r.

$x = r\cos t \qquad x' = -r\sin t$

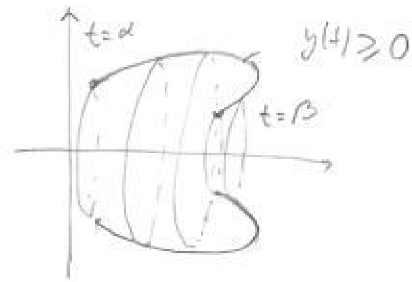
$y = r\sin t \qquad y' = -\cos t$

$0 \le t \le 2\pi$

$$l = \int_0^{2\pi} \sqrt{(r\sin t)^2 + (r\cos t)^2}\, dt = \int_0^{2\pi} r\, dt = 2\pi r$$

Note: If we parametrize the graph of the function $y = f(x)$ in the usual way $\begin{array}{l} x = t \\ y = f(t) \end{array}$ $a \le t \le b$

we get $l = \int_a^b \sqrt{1 + (f'(t))^2}\, dt$ same as old formula.



If parametric curve is rotated about the x-axis the surface obtained has surface area

$$S = \int_\alpha^\beta 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: Surface area of sphere is

$$S = \int_0^\pi 2\pi r \sin t \cdot r\, dt = 4\pi r^2$$