8.4 Taylor polynomials

Def. Suppose $f(x)$ is defined on an open interval I and that all derivatives $f^{(k)}(x)$ exist on I, Fix $a \in I$ and set

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j$$

$T_n(x)$ = n-th Taylor polynomial for f centred at a. If a = 0, we call it the MacLaurin polynomial.

Ex: Let $f(x) = \sqrt{x}$. Find $T_4(x)$, centered at a = 9

j	$f^{(j)}(x)$	$f^{(j)}(9)$
0	$x^{\frac{1}{2}}$	3
1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{4}\cdot\frac{1}{27} = -\frac{1}{108}$
3	$+\frac{3}{8}x^{-\frac{5}{2}}$	$\frac{3}{8}\cdot\frac{1}{343} = \frac{1}{648}$
4	$-\frac{15}{16}x^{-\frac{7}{2}}$	$-\frac{15}{16}\cdot\frac{1}{3^7} = -\frac{5}{11664}$

$$T_4(x) = 3 + \frac{1}{6}(x-3) - \frac{1}{108}(x-3)^2 + \frac{1}{648}(x-3)^3 - \frac{5}{11664}(x-3)^4$$

Ex: Find $T_5(x)$ for $\sin x$, centered at a = 0

j	$f^{(j)}(x)$	$f^{(j)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	−1
4	$\sin x$	0
5	$\cos x$	1

$$T_5(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120}$$

We see from graphs that $T_n(x)$ is close to $\sin x$, especially x is close to 0.

This is in part due to the fact:

$$T_n^{(k)}(a) = f^{(k)}(a) \quad \text{for } k \le n.$$

Let $R_n(x) = f(x) - T_n(x)$.

Theorem: $R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$

$$|R_n(x)| \le K_{n+1} \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

where $K_{n+1} = \max_{t\in[a,x]} |f^{(n+1)}(t)|$

Ex. Accuracy of $T_5(\frac{1}{2})$?
Accuracy of $T_5(x)$ over $(-\frac{\pi}{2}, \frac{\pi}{2})$?

$$K_6 = \max_{t\in[0,\frac{1}{2}]} |-\sin t| = \sin\tfrac{1}{2} \le 1$$

Take $K_6 = 1$

$$|R_n(\tfrac{1}{2})| \le 1 \cdot \frac{(\frac{1}{2}-0)^6}{6!} = \frac{1}{2^6 \cdot 6!} = \frac{1}{46080} = 2.1\times10^{-5}$$

Check: $\sin\frac{1}{2} - T_5(\frac{1}{2}) = -1.5\times10^{-6}$

If we need an estimate of $R_n(x)$ over $[-\frac{\pi}{2}, \frac{\pi}{2}]$

we'll need $K_6 = \max_{t \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |-\sin x| = 1$

$$|R_5(x)| \leq 1 \cdot \frac{|x-0|^6}{6!} \leq \frac{\left(\frac{\pi}{2}\right)^6}{6!} = 2 \times 10^{-2}$$

Ex: How big should n be in order to achieve accuracy 10^{-5} over $[-\frac{\pi}{2}, \frac{\pi}{2}]$?

$$|R_n(x)| \leq 1 \cdot \frac{|x-0|^{n+1}}{(n+1)!} \leq \frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!} \leq 10^{-5}$$

n	$\left(\frac{\pi}{2}\right)^{n+1}/(n+1)!$
5	2×10^{-2}
6	4.7×10^{-3}
7	9.1×10^{-4}
8	1.6×10^{-4}
9	2.5×10^{-5}
10	3.5×10^{-6}

Pf of Theorem:

For $n=2$:

$$f(x) = f(a) + f(x) - f(a)$$
$$= f(a) + \int_a^x f'(t)\,dt = \begin{bmatrix} u = f'(t) & v' = 1 \\ u' = f''(t) & v = t - x \end{bmatrix}$$
$$= f(a) + f'(t)(t-x)\Big|_a^x - \int_a^x f''(t)(t-x)\,dt$$
$$= f(a) - f'(a)(a-x) + \int_a^x f''(t)(x-t)\,dt = \begin{bmatrix} u = f''(t) & v' = (x-t) \\ u' = f'''(t) & v = \frac{1}{2}(x-t)^2(-1) \end{bmatrix}$$
$$= f(a) + f'(a)(x-a) - f''(t)\frac{1}{2}(x-t)^2\Big|_a^x + \int_a^x f'''(t)\frac{(x-t)^2}{2}\,dt$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2}\int_a^x f'''(t)(x-t)^2\,dt$$

etc.