# 11.10. Taylor and Maclaurin series

Suppose $f$ has a series expansion for $|x-a|<R$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots \quad f(a)=c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots \quad f'(a)=c_1$$

$$f''(x) = 2c_2 + 3\cdot 2\cdot c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + \dots \quad f''(a)=2c_2$$

$$f'''(x) = 3\cdot 2\cdot 1\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + \dots \quad f'''(a)=3\cdot 2\cdot 1\, c_3$$

$$f^{(iv)}(x) = 4\cdot 3\cdot 2\cdot 1\, c_4 + \dots \quad f^{(iv)}(a)=4\cdot 3\cdot 2\cdot 1\, c_4$$

$$f^{(n)}(a) = n!\, c_n \quad \text{so} \quad \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Theorem: If $f$ has a power series expansion around $a$, that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ then $c_n = \dfrac{f^{(n)}(a)}{n!}$

Ex: Assume $e^x$, $\sin x$ have a power series expansion around 0. Then

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(state) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Note: odd (even) functions have only odd (even) powers in their expansion.

We can always form a power series for a function just by writing

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

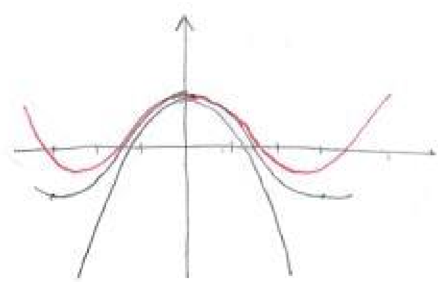
This is called the Taylor series of $f$ at $a$. (Maclaurin series if $a=0$)

The issue is whether the Taylor series converges to the function.

Def. The Taylor polynomial at $a$ of a function $f$ is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Ex. Draw $T_0, T_2, T_4$ for $\cos x$



Let $R_n(x) = f(x) - T_n(x)$

so $f(x) = T_n(x) + R_n(x)$

If $\lim_{n\to\infty} R_n(x) = 0$ when $n\to\infty$ for $|x-a|<R$

then $f(x) =$ sum of its Taylor series.

Theorem: If $|f^{(n+1)}(x)| < M$ for $|x-a| \le d$

then $|R_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$ for $|x-a| \le d$

Note: $\lim_{n\to\infty} \frac{r^n}{n!} = 0$ for every $r$

Ex. $\sin x$ is the sum of its Taylor series.

Can take $M=1$ for every $x$.

For $|x|<d$ we have $|R_n(x)| \le \frac{d^{n+1}}{(n+1)!}$

Then $\lim_{n\to\infty} |R_n(x)| = 0$ by squeeze thm.

So $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for any $|x|<d$

Since this is true for any $d$, then

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for any $x$.

Ex. $\lim_{x\to 0} \frac{\sin x}{x} = 0$ using series

Ex. Find $\sin 20° = \sin \frac{20}{180}\cdot\pi = \sin\frac{\pi}{9}$ with accuracy $10^{-6}$

$\sin\frac{\pi}{9} = \frac{\pi}{9} - \frac{(\frac{\pi}{9})^3}{3!} + \cdots$

$\frac{(\frac{\pi}{9})^{2n+1}}{(2n+1)!}$ is decr.

$n=2 \quad 4\cdot 10^{-5}$

$n=3 \quad 1\cdot 10^{-7}$

$\sin\frac{\pi}{9} \simeq \frac{\pi}{9} - \frac{(\frac{\pi}{9})^3}{3!} + \frac{(\frac{\pi}{9})^5}{5!} = 0.3420202684$

Calc: 0.3420201433

Ex. Compute $\int_0^1 \cos x^2\,dx$ with accuracy $10^{-4}$.

$\int_0^1 \cos x^2\,dx = \int_0^1 \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right)dx$

$= x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \cdots \Big|_0^1$

$= 1 - \frac{1}{5\cdot 2!} + \frac{1}{9\cdot 4!} - \frac{1}{13\cdot 6!} + \cdots$

$\frac{1}{(2n+1)\,n!} < 10^{-4}$ when $(2n+1)\,n! > 10^4$

$n=4 \quad 216$

$n=6 \quad 9360$

$n=8 \quad 685440$

$\int_0^1 \cos x^2\,dx \approx 1 - \frac{1}{5\cdot 2!} + \frac{1}{9\cdot 4!} - \frac{1}{13\cdot 6!}$

$\approx 0.904522792$

Calc gives $\approx 0.90452424$

Ex. Find the first five terms power series for $f(x) = \frac{e^x}{1+x}$

$\frac{e^x}{1+x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - x + x^2 - x^3 + x^4 - \cdots\right)$

$= 1 + (1-1)x + \left(\frac{1}{2!} - 1 + 1\right)x^2 + \left(\frac{1}{3!} - \frac{1}{2!} + 1 - 1\right)x^3$

$+ \left(1 - \frac{1}{3!} + \frac{1}{2!} - 1 + 1\right)x^4 + \cdots$

$= 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3}{2}x^4 - \cdots$