11.9 Representations of functions as power series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{when } |x|<1$$

Ex: $\frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$

if $\left|\frac{x}{3}\right|<1$, $|x|<3$

Ex: $\frac{x^4}{3-x} = \sum_{n=0}^{\infty}\frac{x^{n+4}}{3^{n+1}} \qquad |x|<3$

Ex: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$= 1 - x^2 + x^4 - x^6 + \dots \qquad |x|<1$

$|-x^2|<1$

$x^2<1$

$|x|<1$

Series may be integrated or differentiated term-by-term.

Theorem: Suppose the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$. Then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and continuous on $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x) = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \dots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of conv. for series for f' and $\int f$ are also R.

Ex: $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty}(-1)^n x^n \quad / \frac{d}{dx}$

$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \dots = \sum_{n=0}^{\infty}(-1)^n n x^{n-1}$

$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$

$|x|<1$

Ex: $\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C \qquad |x|<1$

put in $x=0$

$\ln 1 = C$, $C=0$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ for $|x|<1$

If we put in $x=1$

$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

There is a theorem that says we may plug in $x=1$ because $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges (even though the equation was valid for $|x|<1$)

Ex: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \quad / \int dx$

$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad C=0$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad |x|<1$

$x=1 \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}$

($n=50$ $\frac{\pi}{4} \approx 0.78$, accuracy $\frac{1}{101}$

$\frac{\pi}{4} = 0.78539$)