11.8 Power Series

Def. A power series is a series of form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

or, more generally

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

Whether the series converges or not depends on the x we choose; for the x's for which the series converges we get a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Ex: $1 + x + x^2 + \dots = \frac{1}{1-x}$ for $|x|<1$

Note: The partial sum of a power series is a polynomial.

Ex: For which x does $\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$ converge?

Root test: $\sqrt[n]{\left|\frac{x^n}{n5^n}\right|} = \sqrt[n]{\frac{|x|^n}{n5^n}} = \frac{|x|}{5\sqrt[n]{n}} \to \frac{|x|}{5}$

if $|x|<5$ converges

if $|x|>5$ diverges

$x=5$, get $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x=-5$ get $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges

Interval of convergence is $[-5,5)$

Radius of convergence is 5

Ex. Find the interval of convergence for $\sum_{n=2}^{\infty} (\ln n)^n x^n$.

Root test: $\sqrt[n]{|(\ln n)^n x^n|} = |x|(\ln n) \to \infty$ except for $x=0$

Converges only for $x=0$. (radius of convergence is 0)

Ex: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test: $\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{|x|}{n+1} \to 0$ for every x

Interval of conv is $(-\infty, \infty)$, radius is ∞.

Theorem: For the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are three possibilities:

i) series converges only when $x=a$

ii) " " for all x

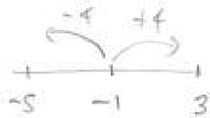
iii) there is a number $R>0$ s.t. series converges if $|x-a|<R$, series diverges if $|x-a|>R$ (when $|x-a|=R$ i.e. when $x=a\pm R$ we have to test separately)

Ex: $\sum_{n=0}^{\infty} \frac{(x+1)^n}{4^{n+3}}$ Find interval of conv.

$$\sqrt[n]{\left|\frac{(x+1)^n}{4^{n+3}}\right|} = \frac{|x+1|}{4\cdot\sqrt[n]{4^3}} \to \frac{|x+1|}{4}$$

$\frac{|x+1|}{4} < 1$ iff $|x+1| < 4$

$|x-(-1)| < 4$



If $x=3$ get $\sum_{n=0}^{\infty} \frac{1}{4^3}$ diverges } by divergence test

$x=-5$ get $\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^3}$ "

Interval of conv. $(-5,3)$, radius $= 4$.