11.5 Alternating Series

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

$-\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots = \sum_{n=1}^{\infty} (-1)^{n} \left(\frac{2}{3}\right)^{n}$

alternating series

Def. A series of type $b_1 - b_2 + b_3 - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n > 0$ is called an alternating series.

Theorem (Alternating series test)
If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

i) $\{b_n\}$ is decreasing

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent. Furthermore, $|R_n| < b_{n+1}$

$|R_n| <$ first ignored term

Proof.

$\{s_{2n}\}$ increasing and bounded: $0 < s_{2n} \le s_1$
so it has a limit s: $\lim_{n\to\infty} s_{2n} = s$

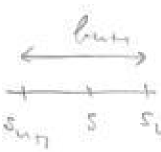
Now $s_{2n+1} = s_{2n} + b_{2n+1}$

$\lim_{n\to\infty} s_{2n+1} = s + 0 = s$

Hence $\{s_n\}$ converges to s, $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = s$

Estimate:

$|s - s_n| \le b_{n+1}$



$|s - s_n| \le b_{n+1}$

Ex. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges

What is the accuracy of s_{10000}?

$|R_{10000}| \le \frac{1}{10000} = 10^{-4}$

$s_{10000} = 0.693097$

$\ln 2 = 0.693147$

Ex. Show that $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$ converges

How many terms are needed to get accuracy 10^{-4}?

$b_n = \frac{1}{(2n)!}$ $\quad |R_n| < \frac{1}{(2(n+1))!}$

$\frac{1}{(2n+2)!} \le 10^{-4}$

$(2n+2)! \ge 10000$

$n = 2 \quad 6! = 720$

$n = 3 \quad 8! = 40320$

$s_3 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} = 0.540278$

$\cos 1 = 0.540302$

11.6 Absolute convergence
Root and Ratio tests

Given $\sum a_n$ we may consider $\sum |a_n| = |a_1| + |a_2| + \dots$

Def. A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ is absolutely convergent

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges but is not absolutely convergent

Def. A series is conditionally convergent if it is convergent, but not absolutely convergent.

Theorem: If $\sum a_n$ is absolutely convergent then it is convergent.

Pf. $0 \le a_n + |a_n| \le 2|a_n|$

so $\sum (a_n + |a_n|)$ converges. Since $\sum |a_n|$ conv,

$$\sum a_n = \sum\big(\underbrace{(a_n + |a_n|)}_{\text{converges}} - \underbrace{a_n}_{\text{converges}}\big) = \sum (a_n + |a_n|) - \sum a_n$$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+n)}{n^2}$ is convergent

$\displaystyle\sum \left| \frac{\sin(n^2+n)}{n^2} \right|$ conv?

$\displaystyle\frac{|\sin(n^2+n)|}{n^2} \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ conv.

Theorem (ratio test) Let $\sum a_n$ be a series.

If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ is absolutely convergent.

If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum a_n$ is divergent.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent

Ex. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{n!}$ is abs. conv., so conv.

Ex. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+1}$ ratio test gives 1, inconclusive.

Theorem (root test) Let $\sum a_n$ be a series

If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum a_n$ is absolutely conv.

If $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ then $\sum a_n$ is divergent.

Ex. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{4^n}$

Note: $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$ $\displaystyle\lim_{n\to\infty} \sqrt[n]{a} = 1$

$\displaystyle\lim_{n\to\infty} \sqrt[n]{P(n)} = 1$

$\displaystyle\lim_{n\to\infty} \sqrt[n]{n^k\left(a_k + \frac{a_{k-1}}{n} + \dots + \frac{a_0}{n^k}\right)} = 1$

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{10^{n+4}}{n^2+17n}$ diverges

Pf of root test:

$\lim \sqrt[n]{|a_n|} = L < 1$

so there exists $L < r < 1$ so that for large n

$$\sqrt[n]{|a_n|} < r < 1$$

Then $|a_n| < r^n$ so $\sum |a_n|$ converges by comparison to geometric series.

Rearrangements:

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} - \frac{1}{4} + \dots = \frac{3}{2} \ln 2$

is a rearrangement of terms of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Theorem: If $\sum a_n$ is abs. convergent, then any rearrangement of $\sum a_n$ has the same sum.

Not true for conditionally conv. series, In fact, those can be rearranged to converge to whatever you want.