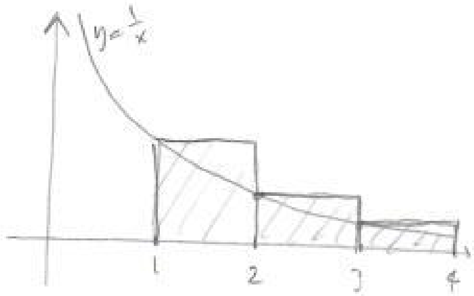
11.3 The integral test

Consider $\sum a_n$, $a_n \geq 0$ for all n.

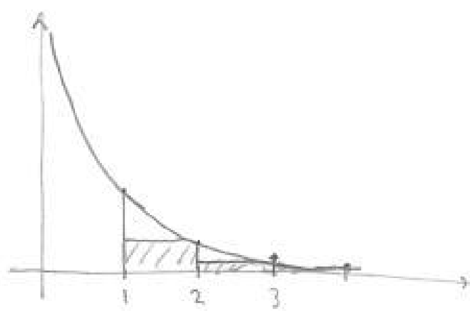
Note that either $\sum a_n$ converges or $\sum a_n = \infty$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$



$\sum_{n=1}^{\infty} \frac{1}{n}$ = area of all rectangles $\geq \int_1^{\infty} \frac{1}{x}\,dx = \infty$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2}$



$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 +$ area of all rectangles $\leq \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx}_{\text{finite}}$

so $\sum \frac{1}{n^2}$ is finite

The integral test: Let f be a continuous, positive decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

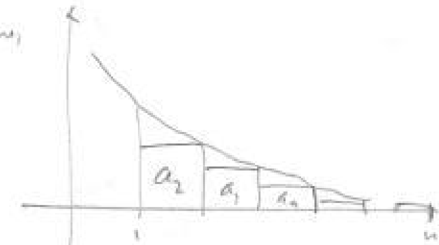
Then:

1) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum a_n$ converges

2) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum a_n$ diverges

Pf of theorem:

1)



$a_2 + \dots + a_n \leq \int_1^{n} f(x)\,dx$

$a_1 + a_2 + \dots + a_n \leq a_1 + \int_1^{n} f(x)\,dx \leq \underbrace{a_1 + \int_1^{\infty} f(x)\,dx}_{= M}$

$S_n \leq M$

and S_n increasing so $\{S_n\}$ has a limit, i.e. $\sum a_n$ converges.

2) similarly.

Ex. $\sum \frac{1}{n^p}$ converges for $p > 1$
diverges for $p \leq 1$

$\sum \frac{1}{n\sqrt{n}}$ converges

$\sum \frac{1}{\sqrt[3]{n}}$ diverges

Suppose $\sum a_n$ converges. Let $s = \sum a_n$

How far is s_n from s?

$$s = a_1 + \dots + a_n + a_{n+1} + \dots$$
$$s_n = a_1 + \dots + a_n$$

$$s - s_n = a_{n+1} + a_{n+2} + \dots$$

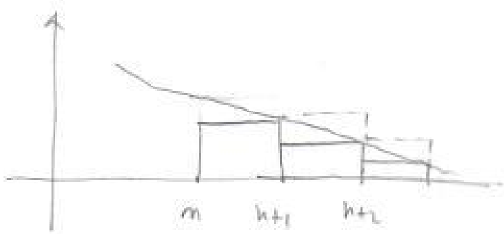
Set $R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$

R_n is the error when approximating the sum by a partial sum.

<u>Theorem</u>: Suppose f is cont., decreasing, positive function on $[1, \infty)$, $a_n = f(n)$. Then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

<u>Pf.</u>



$$\int_{n}^{\infty} f(x)\,dx \le a_{n+1} + a_{n+2} + \dots \le \int_{n}^{\infty} f(x)\,dx$$

<u>Ex</u>: We computed for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ that $s_{10000} = 1.64483$

a) How accurate is this?

b) How many terms do we need to add in order to get accuracy $5 \cdot 10^{-6}$?

a) $R_{10000} \le \int_{10000}^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{10000}^{\infty} = 10^{-4}$

b) $R_n \le \int_{n}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{n}$

$$\frac{1}{n} < 5 \cdot 10^{-6}$$

$$n > \frac{10^6}{5} = 200\,000.$$

$$s_{1000000} = 1.644933$$

11.4 Comparison tests

Ex. Consider $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n}$

$\frac{\sqrt{n}+1}{n} \geq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ Know $\sum \frac{1}{\sqrt{n}}$ diverges

s_n ... nth partial sum of $\sum \frac{\sqrt{n}+1}{n}$

t_n ... nth p $\sum \frac{1}{\sqrt{n}}$

$$\frac{\sqrt{1}+1}{1} + \frac{\sqrt{2}+2}{2} + \dots + \frac{\sqrt{n}+1}{n} \geq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$$

$$s_n \geq t_n$$

We know that $t_n \to \infty$, so $s_n \to \infty$ as well.

Therefore $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n}$ diverges

Ex. Consider $\sum_{n=1}^{\infty} \frac{1}{3^n+7}$.

$\frac{1}{3^n+7} \leq \frac{1}{3^n}$ s_n: nth partial sum of $\sum \frac{1}{3^n+7}$

t_n: ——— $\sum \frac{1}{3^n}$

$$\frac{1}{3^1+7} + \frac{1}{3^2+7} + \dots + \frac{1}{3^n+7} \leq \frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$$

$$s_n \leq t_n$$

Now the limit t_n exists, $t_n \to t$ and $t_n < t$

$\{s_n\}$ is increasing and bounded $s_n < t$

so it converges; i.e. $\sum \frac{1}{3^n+7}$ is conv.

Theorem (Comparison test)

Suppose that $\sum a_n$, $\sum b_n$ are series with positive terms.

i) If $\sum b_n$ converges and $a_n \leq b_n$, then $\sum a_n$ converges

(ii) If $\sum b_n$ diverges and $\underbrace{a_n \geq b_n}_{\text{for all } n,\ \text{enough if true for } n \geq n_0}$ then $\sum a_n$ diverges

Pf. Essentially as at left

smth < finite ⇒ smth is finite

smth > infinite ⇒ smth is infinite

Ex. How to handle something like $\sum \frac{n-3}{2n^2-n}$

$$\frac{n-3}{2n^2-n} < \frac{n}{2n^2-n} \not< \frac{n}{2n^2}$$

bigger denom.

$$\frac{n-3}{2n^2-n} > \frac{n-3}{2n^2} \not> \frac{n}{2n^2}$$

Idea: $\frac{n-3}{2n^2-n} = \frac{n\left(1-\frac{3}{n}\right)}{n^2\left(2-\frac{1}{n}\right)} = \frac{1}{n} \cdot \boxed{\frac{1-\frac{3}{n}}{2-\frac{1}{n}}}$

$\approx \frac{1}{2}$ for large n

For large n $\frac{n-3}{2n^2-n}$ behaves like $\frac{1}{n} \cdot \frac{1}{2}$

We expect that $\sum \frac{n-3}{2n^2-n}$ behaves like $\sum \frac{1}{2n}$

which diverges.

The gist is: $\frac{\frac{n-3}{2n^2-n}}{\frac{1}{n}} = \frac{\frac{1}{n} \cdot \frac{1-\frac{3}{n}}{2-\frac{1}{n}}}{\frac{1}{n}} \longrightarrow \frac{1}{2}$

$\frac{a_n}{b_n} \to \frac{1}{2}$ verifies that $a_n \approx \frac{1}{2} b_n$

Theorem (Limit comparison test)

Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, where $c>0$ and is a finite number, then either both series converge or both diverge.

(i.e. $\sum a_n$ conv. $\Leftrightarrow$ $\sum b_n$ converges)

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3+4n-1}{n^6-n^2+1}$ $\qquad \frac{n^3}{n^6} = \frac{1}{n^3}$ converges

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+n^2}}{3n^{9/2}+n}$ $\qquad \frac{\sqrt{n^4}}{n^{9/2}} = \frac{n^2}{n^{9/2}} = \frac{1}{n^{5/2}}$ converges