12.2 Series

We want to make sense of an infinite sum:

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

0) $1+1+1+\dots$

Ex: a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$

b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n} + \dots$

c) $1 - 1 + 1 - 1 + 1 - 1 + \dots + (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \dots$

d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$

e) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

What does this mean?

Recall: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

Similarly we resolve the infinite sum as follows:

$$1 + \frac{1}{2} + \frac{1}{3} + \dots = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

$$a_1 + a_2 + a_3 + \dots = \lim_{n\to\infty}\left(a_1 + a_2 + \dots + a_n\right)$$

Def: An infinite sum $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is called a series, and is denoted $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

a_n = nth term of the series.

We form a sequence:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3 + \dots$

$\vdots$

$s_n = a_1 + a_2 + a_3 + \dots + a_n$

$\vdots$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ we say the series $\sum a_n$ is convergent and write $s = \sum_{n=1}^{\infty} a_n$.

If $\{s_n\}$ is divergent, then we say $\sum a_n$ is divergent.

Ex. a) $1 + \frac{1}{2} + \frac{1}{3} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$

$s_1 = 1$

$s_2 = \frac{3}{2}$, $s_3 = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$

$s_{10} = 2.92897$

$s_{100} = 5.18738$ appears to diverge

$s_{1000} = 7.48547$

$s_{10000} = 9.78761$

Series divergent (will see) $\left(S_{2^n} > 1 + \frac{n}{2}\right)$

b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$s_1 = 1$

$s_2 = \frac{5}{4}$

$s_3 = \frac{5}{4} + \frac{1}{9} = \frac{49}{36}$

$s_{10} = 1.54977$

$s_{100} = 1.63498$ appears to converge

$s_{1000} = 1.64393$

$s_{10000} = 1.64483$

Series convergent (will see)

c) $1 - 1 + 1 - 1 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1}$

$s_1 = 1$

$s_2 = 0$

$s_3 = 1$

$s_4 = 0$

$\vdots$

$\left.\begin{array}{l} s_{odd} = 1 \\ s_{even} = 0 \end{array}\right\}$ so $\lim_{n\to\infty} s_n$ d.n.e.

$\sum_{n=1}^{\infty} (-1)^{n-1}$ is divergent.

d) $1+\frac{1}{2}+\frac{1}{4}+\dots = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ ← note

This is an example of a geometric series $\sum_{n=0}^{\infty} r^n$.

First: $1+r+r^2+\dots+r^n = \frac{1-r^{n+1}}{1-r}$ (if $r \neq 1$)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^{n+1}$$

$$= \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \text{d.n.e. if } |r|>1 \text{ or } r=-1 \end{cases}$$

If $r=1$ get $\sum_{n=0}^{\infty} 1$, obviously diverges.

$$\sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}$$

Theorem: The geometric series $1+r+r^2+r^3+\dots = \sum_{n=0}^{\infty} r^n$ converges if $|r|<1$ and its sum is $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.
If $|r|\geq 1$ the sum diverges. (slightly different from book)

Ex: $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\dots = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$

Theorem: If $\sum a_n$ and $\sum b_n$ are convergent, then so are $\sum(a_n+b_n)$, $\sum(a_n-b_n)$ and $\sum c a_n$ and their sums are:

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \left(\sum_{n=1}^{\infty} a_n\right) \pm \left(\sum_{n=1}^{\infty} b_n\right)$$

$$\sum_{n=1}^{\infty}(c a_n) = c\left(\sum_{n=1}^{\infty} a_n\right)$$

Note: $\sum_{n=0}^{\infty} a r^n = a\sum_{n=0}^{\infty} r^n = \frac{a}{1-r}$

In general $\sum_{n=k}^{\infty} a(r)^{\text{exp. increasing by 1}} = \frac{\text{First term}}{1-r}$

one way $\frac{\frac{5}{2}}{1-\frac{1}{2}} = 5$

Ex: $\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\dots = \sum_{n=1}^{\infty}\frac{5}{2^n} = \sum_{n=1}^{\infty}\frac{5}{2}\frac{1}{2^{n-1}} = [m=n-1]$

$$= \sum_{m=0}^{\infty}\frac{5}{2}\frac{1}{2^m} = \frac{5}{2}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m = \frac{5}{2}\cdot\frac{1}{1-\frac{1}{2}} = 5$$

Ex: $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{2^{2n}} = \sum_{n=1}^{\infty}\frac{1}{4}\left(\frac{-3}{4}\right)^{n-1} = \frac{\frac{1}{4}}{1-\left(-\frac{3}{4}\right)} = \frac{1}{4}\cdot\frac{4}{7} = \frac{1}{7}$

Ex: $0.2222\ldots = \sum_{n=1}^{\infty}\frac{2}{10^n} = \frac{2}{10}\sum_{m=0}^{\infty}\frac{1}{10^m} = \frac{2}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{2}{9}$

$0.99999\ldots = 1$

Theorem: If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Pf: $a_n = S_n - S_{n-1}$ (illustrate by taking steps: a_n = length of step, S_n = position)

Theorem (test for divergence)
If $\lim_{n\to\infty} a_n \neq 0$ (or d.n.e.) then $\sum a_n$ diverges

Ex: $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

Note: Theorem does not say "if $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges"

Ex: $\sum_{n=1}^{\infty}\frac{1}{n}$

$\lim_{n\to\infty}\frac{1}{n} = 0$, yet $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.