11.1 Sequences

Def. A sequence is an (infinite) list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

first term, second term, n

Notation: $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$ ← may be a different number

Ex. a) $1^2, 2^2, 3^2, \ldots = \{n^2\}_{n=1}^{\infty}$

b) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots = \{\frac{1}{n}\}_{n=1}^{\infty}$

c) $1, -1, 1, -1, \ldots = \{(-1)^n\}_{n=0}^{\infty}$

d) $\{\cos\frac{n\pi}{6}\}_{n=0}^{\infty} = \{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \ldots\}$

e) $\{n\text{-th digit of } \pi\}_{n=1}^{\infty} = 1, 4, 1, 5, 2, 9, \ldots$

f) $\{a_n\}_{n=1}^{\infty}$ where $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$

g) $1.1, 1.01, 1.001, \ldots = \{1 + (\frac{1}{10^n})\}_{n=1}^{\infty}$

Def. We say that the sequence $\{a_n\}$ has a limit L and write $\lim_{n\to\infty} a_n = L$ if we can make a_n arbitrarily close to L by taking n sufficiently large.

If $\lim a_n = L$ we say $\{a_n\}$ converges (diverges otherwise)

Def. $\lim_{n\to\infty} a_n = \infty$ if a_n can be made arbitrarily large by taking n sufficiently large.

Ex. Consider ex. a)-g) and write limits.

Ex. Visualize b), c), d), g)

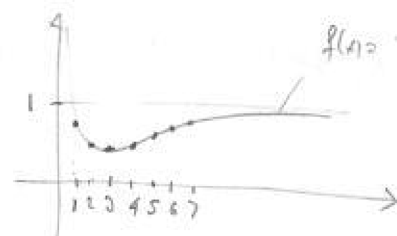
Usually $a_n = f(n)$, e.g. $a_n = \frac{1}{n}$

$a_n = \frac{\ln n}{n^2}$

$a_n = \frac{n^2 - 2n}{2^n}$ etc.

So we may use $f(x)$ in order to examine how $\{a_n\}$ behaves.

Ex. $f(n) = \frac{n^2 + 2n + 3}{n^2 + 8n}$



$\lim_{n\to\infty} a_n = 1$ because $\lim_{x\to\infty} f(x) = 1$

Theorem If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\lim_{n\to\infty} a_n = L$.

Ex. $\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$ (L'Hospital's rule)

Limits of sequences follow the same rules as limits of functions:

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

<u>Theorem:</u> (squeeze theorem) If $a_n \le b_n \le c_n$ for all $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

<u>Ex.</u> $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$$\frac{2\cdot 2\cdot 2\cdot \ldots \cdot 2\cdot 2}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n} \le \frac{2}{1}\cdot \underbrace{\frac{2}{2}\cdot\frac{2}{3}\cdot \ldots \cdot\frac{2}{n-1}}_{\le 1}\cdot\frac{2}{n} \le \frac{4}{n}$$

$$0 \le \frac{2^n}{n!} \le \frac{4}{n} \qquad \lim_{n\to\infty} 0 = 0 \quad \lim_{n\to\infty}\frac{4}{n} = 0$$

By squeeze $\lim_{n\to\infty}\frac{2^n}{n!} = 0$

<u>Theorem</u> If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

<u>Ex.</u> $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

<u>Theorem:</u>
$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{d.n.e.} & \text{if } r\le -1 \end{cases}$$

Pf. If $r>0$ we know
$$\lim_{x\to\infty} r^x = \begin{cases} 0 & \text{if } 0<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \end{cases}$$

so $\lim_{n\to\infty} r^n$ follows same pattern.

If $-1<r<0$ then $\lim_{n\to\infty}|r^n| = 0$ so $\lim_{n\to\infty} r^n = 0$

If $r\le -1$ clearly $\lim r^n$ d...

Def. A sequence is called <u>increasing</u> if $a_n \le a_{n+1}$ for all n, i.e. if $a_1 \le a_2 \le a_3 \le \ldots$

It is called <u>decreasing</u> if $a_1 \ge a_2 \ge a_3 \ge \ldots$.

A sequence is <u>monotonic</u> if it is either increasing or decreasing.

<u>Ex.</u> $\{n^2\}$ is increasing

$\{\frac{1}{n}\}$ is decreasing

<u>Ex.</u> $\frac{n^2}{n^3+1}$ is decreasing for $n \ge 2$

$$\left(\frac{x^2}{x^3+1}\right)' = \frac{2x(x^3+1) - x^2 3x^2}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$2 - x^3 = 0$

$x = \sqrt[3]{2}$

	0		$\sqrt[3]{2}$	
$f'(x)$		$+$		$-$

For $x \ge \sqrt[3]{2}$, function is decreasing

so for $n \ge 2$ sequence is decreasing.

<u>Def.</u> A sequence $\{a_n\}$ is <u>bounded above</u> <u>(below)</u> if there exists a number M so that $a_n \le M$ for all n ($a_n \ge m$ for all n)

A sequence is <u>bounded</u> if it is bounded above and below.

Ex: $0 \le \frac{n^2}{n^2+1} \le 1$ so $\left\{\frac{n^2}{n^2+1}\right\}$ is bounded

Ex: $-1 \le (-1)^n \le 1$ so $\{(-1)^n\}$ is bounded

Ex: $\{n^2\}$ is bounded below, not above

Theorem: Every bounded monotonic sequence is convergent.

nowhere to go

Ex: Let $a_n = \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdots \frac{4n^2-1}{4n^2}$

Clearly $a_n \ge 0$, $a_1 \ge a_2 \ge a_3 \ge \ldots$

Monotonic and bounded, so has a limit

It turns out $\lim_{n \to \infty} a_n = \frac{2}{\pi}$