Mathematical Reasoning — Exam 1
MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) The sky is up and the ground is down.

2. (2pts) For every $x \in \mathbf{R}$, $x^3 > 0$ or $2 + 5 = 7$.

3. (3pts) If $f'(x) = x^2$, then $f(x) = \dfrac{x^3}{3}$.

4. (4pts) (universal set=$\mathbf{Z}$) $x^2 + x < 20$

5. (3pts) (universal set=$\mathbf{R}$) $x^4 + x^2 = 0$

Negate the following statements.

6. (3pts) I am a poor boy and I don't need sympathy.

7. (3pts) If they send me away, then they teach me how to be sensible.

8. (8pts) Use a truth table to prove that $(P \vee Q) \wedge \neg Q \equiv \neg(P \implies Q)$. (Use however many columns you need.)

P	Q								
T	T								
T	F								
F	T								
F	F								

9. (12pts) Use previously proven logical equivalences to prove the equivalence $P \implies (Q \wedge R) \equiv (P \implies Q) \wedge (P \implies R)$. Do not use a truth table.

10. (4pts) Write the converse and contrapositive of the statement: if $x^3 - x - 7 > 0$, then $x > 0$.

Converse:

Contrapositive:

11. (8pts) Suppose the following statements are true:
If the wedding is in China, the bride wears red.
The wedding is in China or the bride wears red.

Determine truth value of the following statement and justify: the bride wears red.

12. (4pts) Use set builder notation to write the set $\{4, 8, 16, 32, 64, \ldots\}$.

13. (10pts) A function $f : \mathbf{R} \to \mathbf{R}$ is *bounded above* if there exists an $M \in \mathbf{R}$ such that for every $x \in \mathbf{R}$, $f(x) \leq M$.
a) Write the definition using symbols for quantifiers.
b) Negate the definition using symbols for quantifiers.
c) Finish the sentence: "A function $f : \mathbf{R} \to \mathbf{R}$ is not bounded above if ..."

14. (7pts) Prove: if m is an even integer, and n is an odd integer, then $m + n$ is an odd integer.

15. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\exists y \in \mathbf{R})(x - y^2 = 3)$$

a) If $x = 0$, is the statement true?
b) If $x = 5$, is the statement true?
c) Find the truth set (the x's) of the above statement.

16. (15pts) We will call an integer n type-0, type-1 or type-2 integer if it can be written in the form $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-2 integer, then $m^2 - n^2 + n$ is a type-2 integer. Start with a know-show table if you find it helpful.

Bonus. (10pts) Show by that quantifiers are not "commutative" by showing that one of the statements below is true and the other is false (justify).

$(\exists y \in \mathbf{R})(\forall x \in \mathbf{R})(x > y)$ $\qquad$ $(\forall x \in \mathbf{R})(\exists y \in \mathbf{R})(x > y)$

Mathematical Reasoning — Exam 2 MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (14pts) Prove using induction: for every natural number n,
$1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

2. (14pts) Prove: for any integer n, the number $\dfrac{2}{3}n^3 + \dfrac{4}{3}n$ is an integer.

3. (16pts) We know that $\sqrt{2}$ is irrational. Are the following statements true? Justify with a counterexample or a proof.

a) There exists a real number x such that $\dfrac{1}{x+\sqrt{2}}$ is rational.

b) If x is rational, then $\dfrac{1}{x+\sqrt{2}}$ is irrational.

4. (18pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $5 \mid a^2 + 2b^2$ if and only if $5 \mid a$ and $5 \mid b$.

a) Write the statement as a conjunction of two conditional statements.

b) Determine whether each of the conditional statements is true, and write a proof, if so.

c) Is the original statement true?

5. (14pts) We have shown a similar statement on homework: for every integer n, if $7 \mid n^2$, then $7 \mid n$. Use this proposition to show that $\sqrt{7}$ is irrational.

6. (10pts) Sketch all points (x, y) in the plane that satisfy $|y - x| < 3$. (Hint: what inequalities without absolute value is the inequality $|u| < a$ equivalent to?)

7. (14pts) Prove both statements for all real numbers x (one is easy):

a) if $x < -1$, then $x + \frac{1}{x+1} < 0$; b) if $x > -1$, then $x + \frac{1}{x+1} \geq 1$.

Bonus. (10pts) Let $0 \leq a_0, a_1, \ldots, a_n \leq 9$ be integers.

a) Use $\pmod 3$ calculus to show that

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Mathematical Reasoning — Exam 3 MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

1. (14pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A \cup B \cup C) - A$ $\qquad$ $B \cap (A \cup C)$ $\qquad$ $A \cap (B - C)$ $\qquad$ $(B - A) \cup (C - A)$

2. (12pts) Let U be the set of real numbers. Consider the intervals of real numbers $A = [7, \infty)$, $B = (7, 9)$, $C = [4, 7]$ and write the following subsets using intervals.

$A \cap C =$

$A \cup C =$

$A^c =$

$A - B =$

$(B \cup C) - A =$

$(C \cup B^c) \cap A =$

3. (14pts) Let $A = \{k \in \mathbf{Z} \mid k \equiv 1 \pmod 3\}$ and $B = \{k \in \mathbf{Z} \mid k \equiv -1 \pmod 3\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $A \cap B = \emptyset$? Prove or disprove.
c) Is $A \cup B = \mathbf{Z}$? Prove or disprove.

4. (16pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = x^2 + 2x \pmod 5$.
a) Write the table of values of f for $x = 0, 1, 2, 3, 4$.
b) Is f injective? Justify.
c) Is f surjective? Justify.
d) Note the domain is $\mathbf{Z}$ (not $\mathbf{Z}_5$). Determine the set of preimages of 4. List at least three elements of this set and describe the set. The table from a) tells you everything you need to know.

5. (12pts) Let $f, g : \mathbf{R} \to \mathbf{R}$, $f(x) = \sqrt{x^2}$, $g(x) = x$.
a) The formulas for f and g are different. Are the functions f and g equal?
b) What is the set of preimages of 3 under f?
c) If f and g are not equal, can you alter the domain or codomain so they are?

6. (10pts) Let $f(x) = \dfrac{3x}{x-2}$ and assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) What is the range of this function? Justify your answer.

7. (10pts) Draw arrow diagrams between two copies of $\mathbf{Z}$ below that illustrate a function $f : \mathbf{Z} \to \mathbf{Z}$ that is:

a) neither a surjection, nor an injection

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

b) a bijection that is not the identity

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

8. (12pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$.

Bonus. (10pts) Let $f : [3, \infty) \to \mathbf{R}$, $f(x) = x^2 - 6x + 8$. The graph of f easily shows that f is injective. Prove injectivity algebraically. *(Hint: difference of squares.)*

Mathematical Reasoning — Final Exam MAT 312, Fall 2020 — D. Ivanšić

Name: ______________________________

Show all your work!

Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is an open sentence, find its truth set.

1. (2pts) If $x \in \mathbf{R}$ and $x^2 < -3$, then $x^2 < -1$.

2. (4pts) (universal set=$\mathbf{R}$) $x^2 - 9x + 14 < 0$.

3. (2pts) For every $x \in \mathbf{R}$, $3x + 7 \geq 0$

Negate the following statements. Write English sentences, but you may assist yourself with symbols, if necessary.

4. (3pts) If the product of two real numbers is positive, then both numbers are positive.

5. (4pts) There exists an $x \in A$, such that x is even and $x \equiv 1 \pmod 3$.

6. (10pts) Use previously proven logical equivalences to prove the equivalence $(P \implies Q) \vee (R \implies Q) \equiv (P \wedge R) \implies Q$. Do not use a truth table.

7. (12pts) Consider the statement: if x is irrational, then $4 + \dfrac{1}{x}$ is irrational.

a) State the converse and prove or disprove it:

b) State the contrapositive and prove or disprove it:

8. (12pts) Let $\mathbf{R}$ be the universal set. The following is an open sentence in x:

$$(\forall y \in \mathbf{R})(y^2 > x)$$

a) If $x = -3$, is the statement true?
b) If $x = 4$, is the statement true?
c) Find the truth set (the x's) of the above statement.

9. (14pts) Prove using induction: for every natural number n,

$$\frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \cdots + \frac{n}{2} = \frac{n(n+1)}{4}$$

10. (12pts) Prove that for all real numbers x and y, $x^2 + 9y^2 \geq 6xy$.

11. (16pts) Consider the statement: for all $a, b \in \mathbf{Z}$, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.
a) Write the statement as a conjunction of two conditional statements.
b) Prove each of the conditional statements.

12. (14pts) Use the statement in problem 11 to show $\sqrt{15}$ is irrational. (Note that a square is a product of two numbers.)

13. (12pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets (shade).
b) Among the three sets, two are equal. Use set algebra to show they are equal.

$A \cap B \cap C$ $\qquad\qquad$ $(C - (C - A)) \cap B$ $\qquad\qquad$ $(A - B) \cap C$

14. (16pts) Let $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$, and let $f : \mathbf{Z} \to \mathbf{Z}_5$, $f(x) = x^2 + 2x \pmod 5$.
a) Write the table of values of f for $x = 0, 1, 2, 3, 4$.
b) Is f injective? Justify.
c) Is f surjective? Justify.
d) Note the domain is $\mathbf{Z}$ (not $\mathbf{Z}_5$). Determine the set of preimages of 4. List at least three elements of this set and describe the set. The table from a) tells you everything you need to know.

15. (5pts) Draw an arrow diagram between the provided two copies of **Z** that illustrates a function $f : \mathbf{Z} \to \mathbf{Z}$ that is is not a surjection and the set of preimages of every element in the codomain has either two or zero elements (pattern needs to be obvious).

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

$\ldots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

16. (12pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$.

Bonus. (10pts) Let $0 \le a_0, a_1, \ldots, a_n \le 9$ be integers.

a) Use $\pmod 3$ calculus to show that

$$a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_n + a_{n-1} + \cdots + a_2 + a_1 + a_0 \pmod 3$$

b) Use a) to show that a natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.